(17,356 words)

Ignorance and polarity sensitivity in the grammar: From disjunction and indefinites to numerals*

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Abstract Given reference to the same domain of individuals/degrees, the pairs *or*/*some* NP_{SG} and a comparative-modified numeral (CMN) / a superlative-modified numeral (SMN) are truth-conditionally equivalent. However, they differ in surprising ways with respect to ignorance and polarity sensitivity, as follows: While all are able to give rise to ignorance, *some* NP_{SG} and CMNs are compatible with positive or negative certainty about a specific member of the domain whereas *or* and SMNs are not. And *some* NP_{SG} and SMNs resist embedding under negation whereas *or* and CMNs do not, although all are able to embed in other downward-entailing environments such as the antecedent of a conditional or the restriction of a universal. There have been many accounts for subsets of these puzzles, but none that would capture the full paradigm for each pair, or the remarkable similarity of the phenomena between the pairs. Drawing on Chierchia (2013)'s alternatives-and-exhaustification approach to epistemic indefinites and polarity sensitivity, I first propose an account for *or*/*some* NP_{SG}, in the process also uncovering the recipe for a general approach to similarity and variation with respect to ignorance and polarity sensitivity. Then, I show that, with a certain new decomposition of CMNs/SMNs, this recipe can be seamlessly extended to them also, in the process also shedding new light on known puzzles to do with their scalar implicatures (and with bare numerals). The overall result is a theory of ignorance and polarity sensitivity fully unified between disjunction/indefinites and modified numerals with welcome consequences for both.

Keywords: disjunction, indefinites, modified numerals, ignorance, positive polarity

1 Introduction

The disjunction *or* and the indefinite *some* NP_{SG} are similar: Given reference to the same domain of individuals, they are truth-conditionally equivalent, (1). However, they differ in surprising ways with respect to ignorance and polarity sensitivity. More concretely, while they can both give rise to speaker ignorance inferences, (2), *some* NP_{SG} is compatible with positive or negative speaker certainty about a specific

* I thank . . .

member of the domain whereas *or* is not, (3)-(4). And *or* can take scope below negation but *some* NP_{SG} cannot, (5), although both are fine in a downward-entailing environment such as the antecedent of a conditional or the restriction of a universal, (6)-(7).

(1) Jo called Alice or Bob / some student. (= 1 *iff* $a \vee b$)

(2) (Who did Jo call?) Jo called Alice or Bob / some student. ($\rightsquigarrow$ ignorance)

(3) Jo called Alice. Therefore, she called # Alice, Bob, or Cindy / ✓some student.

(4) Jo called # Alice, Bob, or Cindy / ✓some student, but not Alice.

(5) Jo didn't call ✓Alice or Bob / # some student.

(6) If Jo called ✓Alice or Bob / ✓some student, she won.

(7) Everyone who called ✓Alice or Bob / ✓some student won.

Strikingly, comparative-modified numerals (CMNs) and superlative-modified numerals (SMNs) exhibit the exact same type of similarity and variation: Given a scale with the same granularity, they are pairwise (e.g., *more than two* / *at least three*, *less than three* / *at most two*) truth-conditionally equivalent, (8). However, while they can both give rise to speaker ignorance inferences, (9), CMNs are compatible with positive or negative speaker certainty about a specific degree in their domain whereas SMNs are not, (10)-(11). And CMNs can take scope below negation but SMNs cannot, (12), although both are fine in downward-entailing environments such as the antecedent of a conditional or the restriction of a universal, (13)-(14).

(8) Jo called less than two people / at most one person. (= 1 *iff* $0 \vee 1$)

(9) (How many people did Jo call?) Jo called less than two people / at most one person. ($\rightsquigarrow$ ignorance)

(10) Jo called two people. Therefore, she called ✓less than three / # at most two.

(11) Jo called ✓less than three / # at most two people, but not one.

(12) Jo didn't call ✓less than two people / # at most one person.

(13) If Jo called ✓less than two people / ✓at most one person, she won.

(14) Everyone who called ✓less than two people / ✓at most one person won.

A global summary of *or*/*some* NP_{SG} and CMNs/SMNs in terms of their points of contrast — compatibility with certainty and anti-negativity — is given in Table 1.

		compatibility with certainty	
		no	**yes**
anti-negativity	**no**	*or*	CMNs
	yes	SMNs	*some* NP_{SG}

Table 1 Compatibility with certainty and anti-negativity in *or*/*some* NP_{SG} and CMNs/SMNs

Subsets of the two paradigms, and even some of the similarities between them, have been recognized and analyzed in the literature (cf., e.g., Sauerland 2004, Meyer 2013 for ignorance in *or*; Spector 2014/Nicolae 2017 for related discussion of ignorance and anti-negativity in the French disjunctions *soit . . . soit*/*ou*; Szabolcsi 2004, Nicolae 2012, a.o., for anti-negativity in *some*; Westera & Brasoveanu 2014, Cremers et al. 2017 for experimental evidence that both CMNs and SMNs can give rise to ignorance and Geurts & Nouwen 2007, Geurts et al. 2010, Cummins & Katsos 2010 for experimental evidence that CMNs are compatible with positive certainty but SMNs are not, or Nouwen et al. 2018 for a recent overview of all these experimental findings; Geurts & Nouwen 2007, Büring 2008, Nouwen 2010, Geurts et al. 2010, Cummins & Katsos 2010, Coppock & Brochhagen 2013, Westera & Brasoveanu 2014, Nouwen 2015, Kennedy 2015, Spector 2015, Mendia 2015, Schwarz 2016, Cremers et al. 2017 for theoretical discussions of ignorance in CMNs and SMNs; Mihoc & Davidson 2017 for experimental evidence of anti-negativity in CMNs but not SMNs, and for similarity in the antecedent of a conditional/restriction of a universal; Nilsen 2007, Geurts & Nouwen 2007, Cohen & Krifka 2014, Spector 2015 for theoretical discussions of anti-negativity in SMNs; Büring 2008, Kennedy 2015 for observations regarding the general similarity between SMNs and disjunction with respect to ignorance and Spector 2014, 2015 for the particular similarity between SMNs and some French disjunctions with respect to both ignorance and polarity sensitivity). However, the complete paradigms, and their remarkable parallelism, have never been recognized or analyzed in full (e.g., most of the existing accounts of CMNs/SMNs do not recognize that CMNs can give rise to ignorance also, neglect the anti-negativity of SMNs, and liken SMNs to disjunction only in the metalanguage). Thus, an account that would capture all the patterns for *or*/*some* NP_{SG}, or all the patterns for CMNs/SMNs, or that would fully explain their similarity, is still missing.

In Section 2 we use Chierchia (2013)'s alternatives-and-exhaustification approach to epistemic indefinites and polarity sensitive items to (a) articulate an account for *or*/*some* NP_{SG} and in the process (b) identify a recipe to derive similarity and variation with respect to ignorance and polarity sensitivity more generally. In Section 3 we (a) use the recipe just developed to articulate an account for CMNs/SMNs and in the process (b) shed new light on a theory of numerals more generally, including bare numerals and scalar implicatures. In Section 4 we summarize our overall results — a theory of ignorance and polarity sensitivity unified for disjunction/indefinites and modified numerals that sheds new light on both disjunction/indefinites and especially numerals — and highlight some open issues — to do with further patterns of immediate relevance, predictions for the range of empirical variation, and predictions for the nature of ungrammaticality.

2 *Or* and *some* NP_{SG}

There are many approaches to ignorance and polarity sensitivity in disjunction and indefinites in the literature. However, all the approaches that derive these phenomena in a *unified* way are approaches based on alternatives and exhaustification (some variant of the grammatical theory of scalar implicatures). And of all these approaches the one that offers the most insights into how to capture *variations* with respect to these phenomena in a unified way is Chierchia (2013). In our search for an account of ignorance and polarity sensitivity in *or*/*some* NP_{SG} we will thus closely follow Chierchia (2013). As we will see, with minor tweaks to the original recipe, we obtain an account that captures all our starting patterns for *or*/*some* NP_{SG}, and gives us a handle on how to capture similarity and variation with respect to ignorance and polarity sensitivity more generally. At the end we compare the resulting theory of ignorance and polarity sensitivity to other existing accounts.

2.1 Truth conditions and alternatives

The first thing we want to capture is the fact that, given reference to the same domain of individuals, *or*/*some* NP_{SG} are truth-conditionally equivalent.

(1) Jo called Alice or Bob / some student. ($= 1$ *iff* $a \vee b$)

This is fairly straightforward. It is known that an *or* utterance can be represented as existential quantification over a domain consisting of its individual disjuncts.[1] And a *some* NP_{SG} utterance can, of course, also be represented as existential quantification over a domain given by the extension of its NP argument.

(15) Jo called *a*, *b* or . . . .
$\exists x \in \{a, b, \ldots\}[C(j, x)]$

1 I put aside here cases with multiple occurrences of the disjunction, where arguably each occurrence can activate alternatives. On those uses *or* is not equivalent to *some* NP_{SG}, although the general approach developed here should still transfer.

(16) Jo called some student.
$\exists x \in [\![\text{student}]\!]\,[C(j,x)]$

If the domain of *or* and of *some* NP_{SG} coincide, their truth-conditional equivalence follows.

Note now that the truth conditions above make reference to both a domain of individuals as well as a scalar element. These pieces naturally give rise to alternatives. In particular, by replacing the domain with its subsets one gets subdomain alternatives, and by replacing the scale with its scalemates one gets scalar alternatives. In fact, by replacing both the domain and the scalar element at the same time one also gets a mixed-type set of alternatives; from these, I will assume that only the ones that are distinct from the existing subdomain and scalar alternatives are retained.

(17) Jo called *a*, *b* or ...

a. $\exists x \in \{a,b,\dots\}[C(j,x)]$ (assertion)
b. $\{\exists x \in D'[C(j,x)] \mid D' \subset \{a,b,\dots\}\}$ (DA)
c. $\{\forall x \in \{a,b,\dots\}[C(j,x)]\}$ (σA)
d. $\{\forall x \in D'[C(j,x)] \mid D' \subset \{a,b,\dots\}\}$ (new, mixed-type DA-σA)

(18) Jo called some student.

a. $\exists x \in [\![\text{student}]\!]\,[C(j,x)]$ (assertion)
b. $\{\exists x \in D'[C(j,x)] \mid D' \subset [\![\text{student}]\!]\}$ (DA)
c. $\{\forall x \in [\![\text{student}]\!]\,[C(j,x)]\}$ (σA)
d. $\{\forall x \in D'[C(j,x)] \mid D' \subset [\![\text{student}]\!]\}$ (new, mixed-type DA-σA)

Putting the results above together, note that, given reference to the same domain of individuals, *or* and *some* NP_{SG} yield not just identical truth conditions but also identical subdomain and scalar alternatives. For example, *or*/*some* NP_{SG} for the same two-element domain have the truth conditions and alternatives below, henceforth labeled and abbreviated as shown on the right:

(19) Jo called Alice or Bob / some student$_{[\![\text{student}]\!]=\{a,b\}}$.

a. $\exists x \in \{a,b\}[C(j,x)]$ (assertion; abbr. $a \vee b$)
b. $\exists x \in \{a\}[C(j,x)]$ (singleton DA; abbr. a)
   $\exists x \in \{b\}[C(j,x)]$ (singleton DA; abbr. b)
c. $\forall x \in \{a,b\}[C(j,x)]$ (σA; abbr. $a \wedge b$)
d. — (no DA-σA different from existing DA, σA)

And *or*/*some* NP_{SG} for the same three-element domain have the truth conditions and alternatives below, henceforth abbreviated as shown on the right:

(20) Jo called Alice, Bob, or Cindy / some student$_{[\![\text{student}]\!]=\{a,b,c\}}$.

a. $\exists x \in \{a,b,c\}[C(j,x)]$ (assertion; abbr. $a \vee b \vee c$)
b. $\exists x \in \{a\}[C(j,x)]$ (singleton DA; abbr. a)
$\exists x \in \{b\}[C(j,x)]$ (singleton DA; abbr. b)
$\exists x \in \{c\}[C(j,x)]$ (singleton DA; abbr. c)
$\exists x \in \{a,b\}[C(j,x)]$ (doubleton DA; abbr. $a \vee b$)
$\exists x \in \{a,c\}[C(j,x)]$ (doubleton DA; abbr. $a \vee c$)
$\exists x \in \{b,c\}[C(j,x)]$ (doubleton DA; abbr. $b \vee c$)
c. $\forall x \in \{a,b,c\}[C(j,x)]$ (σA; abbr. $a \wedge b \wedge c$)
d. $\forall x \in \{a,b\}[C(j,x)]$ (doubleton DA-σA; abbr. $a \wedge b$)
$\forall x \in \{a,c\}[C(j,x)]$ (doubleton DA-σA; abbr. $a \wedge c$)
$\forall x \in \{b,c\}[C(j,x)]$ (doubleton DA-σA; abbr. $b \wedge c$)

2.2 Exhaustification

Chierchia (2013) proposes that, syntactically, items like *or* and *some* NP_{SG} that naturally give rise to subdomain and scalar alternatives can be seen as carrying unvalued subdomain and scalar alternative features. This triggers the insertion at a c-commanding position of a silent alternative-sensitive operator O, which checks off those features as shown below. For example, an insertion of $\mathrm{O_{DA}}/\mathrm{O_{\sigma A}}$ corresponds to the domain/scalar feature getting a plus (and to the other feature getting a minus).

(21) $\mathrm{O_{DA}}$(Jo called Alice or$_{[-\sigma,+\mathrm{D}]}$ Bob / some$_{[-\sigma,+\mathrm{D}]}$ student.)

(22) $\mathrm{O_{\sigma A}}$(Jo called Alice or$_{[+\sigma,-\mathrm{D}]}$ Bob / some$_{[+\sigma,-\mathrm{D}]}$ student.)

Semantically, this valuation of an alternative feature corresponds to the activation of the alternatives corresponding to that feature. The alternatives thus activated grow as in Rooth (1985)-style alternative semantics until they meet the operator O, which factors them into meaning, as follows: Given p, a proposition, and $[\![\mathrm{p}]\!]^{\mathrm{C}}$, a set of alternatives C to p, an application of $\mathrm{O_C}$ to p (aka the prejacent of O) will assert p and furthermore say that all the propositions in $[\![\mathrm{p}]\!]^{\mathrm{C}}$ that are true are already entailed by p, (23) (Chierchia 2013: 139), that is, that all of the non-entailed (stronger or logically independent) alternatives to p are false.

(23) $[\![\mathrm{O_C}(p)]\!]^{g,w} = [\![\mathrm{p}]\!]^{g,w} \wedge \forall q \in [\![\mathrm{p}]\!]^{\mathrm{C}}\ [[\![\mathrm{q}]\!]^{g,w} \rightarrow \lambda w' .\ [\![\mathrm{p}]\!]^{g,w'} \subseteq q]$

For example, $\mathrm{O_{DA}}/\mathrm{O_{\sigma A}}$ (whose meaning is obtained by replacing C above with DA/σA) applied to $(p \vee q)$ asserts $(p \vee q)$ and negates its non-entailed DA/σA, the result being a contradiction (of the G(rammatically)-trivial kind, a reason for ungrammaticality) (cf. Chierchia 2013: 51 and references therein) / a traditional scalar implicature.

(24) $\mathrm{O}_{\mathrm{DA}}(a \vee b) = (a \vee b) \wedge \neg a \wedge \neg b, \ = \bot$ (G-trivial)

(25) $\mathrm{O}_{\sigma\mathrm{A}}(a \vee b) = (a \vee b) \wedge \neg(a \wedge b)$ ($\rightsquigarrow$ *not and/every*)

Chierchia argues that items may vary in whether their subdomain alternatives generated as above are in fact as derived, that is, as DA, or in a pre-exhaustified form, that is, as ExhDA. An ExhDA is essentially a fully grown DA prefixed by O (though see Section 2.5) and interpreted exhaustively relative to all the other DA in the DA set (to be revised).

(26) $[\![\mathrm{p}]\!]^{\mathrm{ExhDA}} = \{\mathrm{O}_{\mathrm{DA}}(q) :\ q \in [\![\mathrm{p}]\!]^{\mathrm{DA}}\}$

I will assume that *or* and *some* NP_{SG} are items whose subdomain alternatives must be used in their pre-exhaustified form. That is, the subdomain alternatives that are relevant for their interpretation are not directly the DA generated through replacement of the domain with its subsets, but rather their variants prefixed by O. Thus, their domain feature is checked off not via O_{DA} but rather via $\mathrm{O}_{\mathrm{ExhDA}}$.

In fact, Chierchia also argues that items may also vary in whether all their alternative features / alternatives must be checked off / activated or not. I will assume that the pre-exhaustified subdomain alternatives and scalar alternatives of *or*/*some* NP_{SG} are both factored in by default via insertion of both $\mathrm{O}_{\mathrm{ExhDA}}$ and $\mathrm{O}_{\sigma\mathrm{A}}$. This can be done at different sites or at the same site, factored sequentially as $\mathrm{O}_{\mathrm{ExhDA}}(\mathrm{O}_{\sigma\mathrm{A}}(a \vee b))$ or $\mathrm{O}_{\sigma\mathrm{A}}(\mathrm{O}_{\mathrm{ExhDA}}(a \vee b))$ or together as $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}$; for most of our cases of interest the choice will not make a difference, so for brevity and legibility we will simply use $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}$, as shown below.[2]

(27) $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}$(Jo called Alice or$_{[+\sigma,+\mathrm{D}]}$ Bob / some$_{[+\sigma,+\mathrm{D}]}$ student.)

This assumption that *or*/*some* NP_{SG} must be exhaustified relative to ExhDA, and that in fact their σA are also factored in by default, is the first piece in our analysis of their ignorance and polarity sensitivity patterns.

2.3 Ignorance

We said that *or*/*some* NP_{SG} are such that both their domain and their scalar feature must be checked off, as for example shown above in (27). (Below and going forward we skip explicit marking of this step.) That means that they must undergo exhaustification relative to both ExhDA and σA, that is, in the simplest version, they must be in the scope of $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}$. $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}$ (whose meaning is obtained as above for

2 Regarding those alternative parses, I am assuming that the domain alternatives of $\mathrm{O}_{\sigma\mathrm{A}}(a \vee b)$ are the same as those of $(a \vee b)$ — the idea being that $\mathrm{O}_{\sigma\mathrm{A}}(a)$ is undefined, that is, a domain alternative doesn't have its own scalar alternative — and the scalar alternative of $\mathrm{O}_{\mathrm{ExhDA}}(a \vee b)$ is $\mathrm{O}_{\mathrm{ExhDA}}(a \wedge b)$.

O_{DA}, $O_{\sigma A}$) assserts the prejacent, (28-a), negates the non-entailed ExhDA, (28-b), and negates the non-entailed σA, (28-c) (below and going forward we will always list these on separate lines, for clarity).

(28) Jo called Alice or Bob / some student.
$O_{\text{ExhDA}+\sigma\text{A}}(a \vee b)$
a. $(a \vee b) \wedge$ (prejacent)
b. $\underbrace{\neg \underbrace{\mathrm{O}a}_{a\wedge\neg b}}_{a\rightarrow b} \wedge \underbrace{\neg \underbrace{\mathrm{O}b}_{b\wedge\neg a}}_{b\rightarrow a} \wedge$ (ExhDA-implicatures)
c. $\neg(a \wedge b)$ (σA-implicature)
$= \bot$ (G-trivial)

Note that, given the logical form of each ExhDA, their negations amount to a series of implications. By the meaning of logical implication, these implications can be true together *iff* $\neg a \wedge \neg b$ or $a \wedge b$. Of these, only the latter solution is consistent with the prejacent, so the resulting ExhDA-implicature is $a \wedge b$. However, this meaning clashes with the σA-implicature $\neg(a \wedge b)$, the overall result being a contradiction.

But if the utterance above must be exhaustified relative to both ExhDA and σA, and if this yields a contradiction, how then do we explain our starting claim, repeated below, that this utterance is grammatical, and furthermore yields ignorance?

(2) (Who did Jo call?) Jo called Alice or Bob / some student. ($\rightsquigarrow$ ignorance)

Intuitions about ignorance are uniformly that it is a silent modal type of meaning. To see how this silent modal meaning might arise from $O_{\text{ExhDA}+\sigma\text{A}}$, it is first useful to see what $O_{\text{ExhDA}+\sigma\text{A}}$ yields for overt modal meanings.

Consider first $O_{\text{ExhDA}+\sigma\text{A}}$ across an overt possibility modal.

(29) Jo may call Alice or Bob / some student.
$O_{\text{ExhDA}+\sigma\text{A}}(\Diamond(a \vee b))$
a. $\Diamond(a \vee b) \wedge$
b. $\underbrace{\neg \underbrace{\mathrm{O}(\Diamond a)}_{\Diamond a\wedge\neg\Diamond b}}_{\Diamond a\rightarrow\Diamond b} \wedge \underbrace{\neg \underbrace{\mathrm{O}(\Diamond b)}_{\Diamond b\wedge\neg\Diamond a}}_{\Diamond b\rightarrow\Diamond a} \wedge$
c. $\neg\Diamond(a \wedge b)$
$= \Diamond(a \vee b) \wedge \Diamond a \wedge \Diamond b \wedge \neg\Diamond(a \wedge b)$
'There is an accessible world where Jo calls Alice or Bob and there is an accessible world where she calls Alice, and there is an accessible world where she calls Bob, and there is no accessible world where she calls both.'

The implications arising from the ExhDA are consistent with $\neg\Diamond a \wedge \neg\Diamond b$ or with $\Diamond a \wedge \Diamond b$, but only the latter meaning is consistent with the prejacent. The ExhDA-implicatures thus amount to a Free Choice effect (the well-known Free Choice effect of disjunction under a possibility modal), further strengthened by the σA-implicature.

Consider now $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}$ across an overt necessity modal.

(30) Jo must call Alice or Bob / some student.
$\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}(\Box(a \vee b))$
a. $\Box(a \vee b) \wedge$
b. $\underbrace{\neg \underbrace{\mathrm{O}(\Box a)}_{\Box a \wedge \neg \Box b}}_{\Box a \rightarrow \Box b} \wedge \underbrace{\neg \underbrace{\mathrm{O}(\Box b)}_{\Box b \wedge \neg \Box a}}_{\Box b \rightarrow \Box a} \wedge$
c. $\neg\Box(a \wedge b)$
$= \underbrace{\Box(a \vee b) \wedge \neg\Box a \wedge \neg\Box b}_{\Box(a \vee b) \wedge \Diamond a \wedge \Diamond b} \wedge \neg\Box(a \wedge b)$
'In every accessible world Jo calls Alice or Bob, and it is not the case that in every world she calls Alice, and it is not the case that in every world she calls Bob, and it is not the case that in every world she calls both.'

The implications arising from the ExhDA are consistent with $\neg\Box a \wedge \neg\Box b$ or with $\Box a \wedge \Box b$, and both are consistent with the prejacent, but only the former is consistent with the σA-implicature $\neg\Box(a \wedge b)$. The result is again a Free Choice effect plus a σA-implicature.

With Grice (1975), Sauerland (2004), Meyer (2013), Chierchia (2013), Kratzer & Shimoyama (2017 [2002]) and many others, I will assume that ignorance in seemingly episodic contexts arises because these contexts are in fact not episodic but rather contain a silent, matrix-level, speaker-oriented epistemic necessity modal, let's call it $\Box_S$ ('null speaker-oriented epistemic Box'). That is, our starting *or/some NP$_{SG}$* utterance is not $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}(p \vee q)$ but rather $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}\Box_S(p \vee q)$, as below.

(31) Jo called Alice or Bob / some student.
$\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}(\Box_S(a \vee b))$
a. $\Box_S(a \vee b) \wedge$
b. $\underbrace{\neg \underbrace{\mathrm{O}(\Box_S a)}_{\Box_S a \wedge \neg \Box_S b}}_{\Box_S a \rightarrow \Box_S b} \wedge \underbrace{\neg \underbrace{\mathrm{O}(\Box_S b)}_{\Box_S b \wedge \neg \Box_S a}}_{\Box_S b \rightarrow \Box_S a} \wedge$
c. $\neg\Box_S(a \wedge b)$

$$= \underbrace{\Box_S(a \vee b) \wedge \neg\Box_S a \wedge \neg\Box_S b}_{\Box_S(a\vee b)\wedge\Diamond_S a\wedge\Diamond_S b} \wedge \neg\Box_S(a \wedge b)$$

'In every accessible world (world compatible with what the speaker knows) Jo calls Alice or Bob, and it is not the case that in every world she calls Alice, and it is not the case that in every world she calls Bob, and it is not the case that in every world she calls both.' ($\rightsquigarrow$ ignorance)

Just as for the overt necessity modal case, the implications arising from the ExhDA are consistent with $\neg\Box_S a \wedge \neg\Box_S b$ or with $\Box_S a \wedge \Box_S b$, and both these meanings are consistent with the prejacent, but only the former is consistent with the σA-implicature $\neg\Box(a \wedge b)$. The result is again a Free Choice effect plus a σA-implicature. However, due to the nature of the modal, this time it is a speaker-oriented, epistemic Free Choice effect, that is, ignorance. This captures (2). (With Chierchia (2013)/Kratzer & Shimoyama (2017 [2002]), I assume that this null modal is inserted as a last resort mechanism to rescue an exhaustification parse that would otherwise crash, and that this modal may be of other kinds also, for example, an agent-oriented bouletic necessity modal, yielding indifference rather than ignorance.)

But the ignorance effect we obtained was total — the speaker isn't certain about any of the individuals in the domain. How then do we capture the facts, repeated below, that in addition to being able to give rise to ignorance, *some* NP_{SG} but not *or* is also compatible with positive/negative certainty?

(3) Jo called Alice. Therefore, she called # Alice, Bob, or Cindy / ✓some student.

(4) Jo called # Alice, Bob, or Cindy / ✓some student, but not Alice.

Discussing variation similar to ours, Chierchia (2013) argues that it arises when an item can be exhaustified relative to just a natural subset of its DA, for example, just the singleton or just the non-singleton DA. We will adopt this basic line also. Now, the examples we have discussed so far only have singleton DA. Removing them would thus leave us with no alternatives. We will assume that such an option is never allowed as it would destroy the domain. The minimal case that we must discuss is then, as in our own starting examples, a case with a 3-element domain, that is, a case where there are both singleton and non-singleton DA.

Below we discuss three different exhaustifications for a 3-element domain, one relative to just the singleton DA, one relative to just the non-singleton (doubleton) DA, and one relative to the full set of DA (for a refresher on the alternatives, see (20) above). The results are evaluated as before, by looking for a meaning compatible with all of the prejacent, the negations of the non-entailed ExhDA, and the negations of the non-entailed σA. However, whereas before there was just one possible result, in these cases there are multiple. To guide our assessment and presentation, we will throughout consider compatibility with four possible models of interest: (M1) no ignorance / ‘all winners’; (M2) partial ignorance with positive certainty / ‘one winner’; (M3) partial ignorance with negative certainty /‘one loser’; and (M4) total ignorance / ‘no winner’. (The first set of labels reflects our descriptions of the patterns of interest so far, and the second set describes the situation for the DA and is modeled on more general labels used in the literature used to describe variation effects uniformly across cases with possibility and necessity modals, cf. Chierchia 2013 and references therein.)[3]

First, consider an exhaustification relative to pre-exhaustified *singleton* DA and σA, $O_{\text{ExhSgDA}+\sigma\text{A}}$. Assume that pre-exhaustification of each SgDA happens relative to all the other SgDA.

(32) $O_{\text{ExhSgDA}+\sigma\text{A}}\Box_S(a \vee b \vee c)$

a. $\Box_S(a \vee b \vee c) \wedge$

b. $\underbrace{\neg \underbrace{O\Box_S a}_{\Box_S a \wedge \neg\Box_S b \wedge \neg\Box_S c}}_{\Box_S a \rightarrow \Box_S b \vee \Box_S c} \wedge \underbrace{\neg \underbrace{O\Box_S b}_{\Box_S b \wedge \neg\Box_S a \wedge \neg\Box_S c}}_{\Box_S b \rightarrow \Box_S a \vee \Box_S c} \wedge \underbrace{\neg \underbrace{O\Box_S c}_{\Box_S c \wedge \neg\Box_S a \wedge \neg\Box_S b}}_{\Box_S c \rightarrow \Box_S a \vee \Box_S b} \wedge$

c. $\neg\Box_S(a \wedge b \wedge c)$

(M1) no ignorance / ‘all winners’:
$\Box_S a \wedge \Box_S b \wedge \Box_S c$ ✗/✓
(Clash with the σA-implicature. Possible if it is suspended.)

(M2) partial ignorance with positive certainty / ‘one winner’:
$\Box_S a \wedge \neg\Box_S / \Box_S \neg b \wedge \neg\Box_S / \Box_S \neg c$ ✗
(Suppose $\Box_S a$. Then, if $\neg\Box_S b$ is true and $\neg\Box_S c$ is true, the second and the third implication can be true, but the first one cannot.)

(M3) partial ignorance with negative certainty / ‘one loser’:
$\Box_S \neg a \wedge \neg\Box_S b \wedge \neg\Box_S c$ ✓

(M4) total ignorance / ‘no winner’:
$\neg\Box_S a \wedge \neg\Box_S b \wedge \neg\Box_S c$ ✓

To sum up, $O_{\text{ExhSgDA}+\sigma\text{A}}$ can be verified by a model of partial ignorance of the negative certainty ‘one loser’ type or by a model of total ignorance. It can also be verified by a model of no ignorance if the σA-implicatures are suspended.

3 For example, the ‘one loser’ label accurately describes both $\Box_S\neg a \wedge \neg\Box_S b \wedge \neg\Box_S c$ and $\neg\Diamond a \wedge \Diamond b \wedge \Diamond c$, and the ‘no loser’ label accurately describes both $\neg\Box_S a \wedge \neg\Box_S b \wedge \neg\Box_S c$ and $\Diamond_S a \wedge \Diamond_S b \wedge \Diamond_S c$. We won’t be able to discuss variation with possibility modals here other than briefly in Section 2.5, but mention this in order to facilitate future discussion.

Second, consider exhaustification relative to pre-exhaustified *non-singleton* DA and σA, $O_{\text{ExhNonSgDA}+\sigma\text{A}}$. Assume that pre-exhaustification of each NonSgDA happens relative to all the other NonSgDA.

(33)
$O_{\text{ExhNonSgDA}+\sigma\text{A}}\Box_S(a \lor b \lor c)$
a. $\Box_S(a \lor b \lor c) \land$
b. $\underbrace{\neg \underbrace{O\Box_S(a \lor b)}_{\Box_S(a\lor b)\land\neg\Box_S(a\lor c)\land\neg\Box_S(b\lor c)}}_{\Box_S(a\lor b)\to\Box_S(a\lor c)\lor\Box_S(b\lor c)} \land \underbrace{\neg \underbrace{O\Box_S(a \lor c)}_{\Box_S(a\lor c)\land\neg\Box_S(a\lor b)\land\neg\Box_S(b\lor c)}}_{\Box_S(a\lor c)\to\Box_S(a\lor b)\lor\Box_S(b\lor c)} \land \underbrace{\neg \underbrace{O\Box_S(b \lor c)}_{\Box_S(b\lor c)\land\neg\Box_S(a\lor b)\land\neg\Box_S(a\lor c)}}_{\Box_S(b\lor c)\to\Box_S(a\lor b)\lor\Box_S(a\lor c)} \land$
c. $\neg\Box_S(a \land b) \land \neg\Box_S(a \land c) \land \neg\Box_S(b \land c) \land \neg\Box_S(a \land b \land c)$

(M1) no ignorance / 'all winners':
$\Box_S a \land \Box_S b \land \Box_S c$ ✗/✓
(Clash with the σA-implicatures. Possible if they are suspended.)

(M2) partial ignorance with positive certainty / 'one winner':
$\Box_S a \land \neg\Box_S/\Box_S\neg b \land \neg\Box_S/\Box_S\neg c$ ✓

(M3) partial ignorance with negative certainty /'one loser':
$\Box_S\neg a \land \neg\Box_S b \land \neg\Box_S c$ ✗
(Consider, for example, the third implication. Suppose $\Box_S\neg a$ is true. Then, if $\neg\Box_S b \land \neg\Box_S c$ is also true, the whole consequent is false. This means that the implication can be true iff the antecedent $\Box_S(b \lor c)$ is also false. But this would contradict $\Box_S(a \lor b \lor c) \land \Box_S\neg a = \Box_S(b \lor c)$.)

(M4) total ignorance / 'no winner':
$\neg\Box_S a \land \neg\Box_S b \land \neg\Box_S c$ ✓

To sum up, $O_{\text{ExhNonSgDA}+\sigma\text{A}}$ can be verified by a model of partial ignorance of the positive certainty 'one winner' type or by a modal of total ignorance. It can also be verified by a modal of no ignorance if the σA-implicatures are suspended.

Third, and last, consider exhaustification relative to pre-exhaustified DA (that is, the full set of DA, both singleton and non-singleton) and σA, $O_{\text{ExhDA}+\sigma\text{A}}$. If we continue to assume that pre-exhaustification of the SgDA and NonSgDA happens as before, that is, relative to all the other SgDA and NonSgDA, respectively, the result in this case is simply the intersection of the results we obtained in the $O_{\text{ExhSgDA}+\sigma\text{A}}$ and $O_{\text{ExhNonSgDA}+\sigma\text{A}}$ case, that is, total ignorance or, if the σA-implicatures are suspended, no ignorance.

The solution to the difference with respect to the strength of the ignorance effect in *or* and *some NP_{SG}* that we will propose is then as follows: By default both *or* and *some* NP_{SG} have to be exhaustified relative to the full set of ExhDA and σA, and the result is as in the no pruning case, that is, total ignorance. In the presence of a context of partial ignorance of the 'one winner' or 'one loser' type, *some* NP_{SG} but not *or* is able to also prune its DA set to either just the non-singleton DA — this accommodates the 'one winner' case — or just the singleton DA — this accommodates the 'one loser' case. We have now captured (3)-(4).

Note that, due to the similar logical shape, this view predicts parallel total and partial variation effects under an overt necessity modal or in the scope of a universal quantifier also. This prediction seems to be borne out. (One difference is that variation, whether total or partial, can be of two types, coming from $O_{\text{ExhDA}+\sigma\text{A}} > \Box/\forall > \vee$ or $O_{\text{ExhDA}+\sigma\text{A}} > \Box_S > \vee > \Box/\forall$; the latter is ignorance.)

(34) a. Jo must call Alice, Bob, or Cindy / some student. ($\rightsquigarrow$ variability)
b. Jo must call Alice. Therefore, she must call # Alice, Bob, or Cindy / ✓some student.
c. Jo must call # Alice, Bob, or Cindy / ✓some student, but not Alice.

(35) a. Everyone called Alice, Bob, or Cindy / some student. ($\rightsquigarrow$ variability)
b. Everyone called Alice. Therefore, everyone called # Alice, Bob, or Cindy / ✓some student.
c. Everyone called # Alice, Bob, or Cindy / ✓some student, but not Alice.

Finally, we kept track in each case of the fact that, if the σA-implicatures are suspended, the result is also compatible with no ignorance. This is borne out: an *or*/*some* NP_{SG} utterance can be followed up with *in fact, both/every*.

(36) Jo called Alice or Bob / some student. In fact, she called both Alice and Bob / every student.

(37) Jo must call Alice or Bob / some student. In fact, she must call both Alice and Bob / every student.

(38) Everyone called Alice or Bob / some student. In fact, everyone called both Alice and Bob / every student.

Yet if this suspension of the σA-implicatures were available freely, we would predict that *or*/*some* NP_{SG} would as a default be able to mean *or*/*every*. This result seems too strong. I propose that, at least for *or*/*some* NP_{SG}, the suspension of the σA-implicatures / pruning of the σA happens just like the pruning of the DA (when possible), that is, only as a way to accommodate an otherwise contradictory context.

2.4 Polarity sensitivity

Earlier we captured the ignorance patterns of *or/some* NP_{SG}. Now let us turn to their polarity sensitivity patterns, repeated below, which again include both a contrast and a similarity. How can we capture these patterns?

(5) Jo didn't call ✓Alice, Bob, or Cindy / # some student.

(6) If Jo called ✓Alice, Bob, or Cindy / ✓some student, she won.

(7) Everyone who called ✓Alice, Bob, or Cindy / ✓some student won.

Let's first note what happens when we exhaustify *or/some* NP_{SG} across negation.

(39) $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}(\neg(a \vee b))$

a. $\neg(a \vee b)$

b. $\neg \underbrace{\mathrm{O}(\neg a)}_{\substack{\neg a \wedge \neg\neg b,\ = \neg a \wedge b \\ \text{already excluded by the prejacent}}} \wedge \neg \underbrace{\mathrm{O}(\neg b)}_{\substack{\neg b \wedge \neg\neg a,\ = \neg b \wedge a \\ \text{already excluded by the prejacent}}}$

c. $\neg \underbrace{(\neg(a \wedge b))}_{\text{already entailed by the prejacent}}$

$= \neg(a \vee b)$

Each of the ExhDA is incompatible with the prejacent, therefore already excluded by it, so negating it doesn't lead to any strengthening. And the σA is itself entailed by the prejacent so it cannot actually be negated. The overall result is thus vacuous.

Now, if we exhaustify *or/some* NP_{SG} across *if/every* we should get the exact same result as when exhaustifying across *not*, since the results above are based on the downward monotonicity of the environments. But should these exhaustifications actually proceed in the same way?

It has been known since von Fintel (1999) that, unlike the scope of negation, downward-entailing environments such as the antecedent of a conditional and the restriction of a universal carry an existential presupposition.

(40) Jo didn't call Alice or Bob / # some student.
presupposes: nothing

(41) If Jo called Alice or Bob / some student, she won.
presupposes: There is an accessible world where Jo called Alice or Bob / some student.

(42) Everyone who called Alice or Bob / some student won.
presupposes: There is someone who called Alice or Bob / some student.

And it has been known since Gajewski (2011) that polarity sensitive items may be sensitive to these presuppositions. Building on this, Chierchia (2013) argues that for some items exhaustification may proceed in a strong way, relative to the presupposition-enriched assertion and its presupposition-enriched alternatives. The presupposition-enriched content of a proposition p, $\pi(p)$, and strong exhaustification in terms of $\pi(p)$, O^{S}, are defined as below (Chierchia 2013: 219-20).

(43) $\pi(p) = {}^{\alpha}p \wedge {}^{\pi}p$
where ${}^{\alpha/\pi}p$ = the assertive/presuppositional component of p

(44) $[\![\mathrm{O}^{\mathrm{S}}_{\mathrm{C}}(p)]\!]^{g,w} = [\![\mathrm{p}]\!]^{g,w} \wedge \forall q \in [\![\mathrm{p}]\!]^{\mathrm{C}}\ [\pi([\![\mathrm{q}]\!])^{g,w} \rightarrow \pi(\lambda w'.\, [\![\mathrm{p}]\!]^{g,w'}) \subseteq \pi([\![\mathrm{q}]\!])]$

Taking these into account, suppose that the exhaustification of *or*/*some* NP_{SG} across *if*/*every* must actually proceed relative to presupposition-enriched content, that is, via $\mathrm{O}^{\mathrm{S}}_{\mathrm{ExhDA+\sigma A}}$. Thus, it has the shape below (where v = the world/individual variable w/x from the conditional / universal, a = 'Jo called Alice / 'called Alice', and b = 'Jo called Bob / 'called Bob'), with the result below.

(45) $\mathrm{O}^{\mathrm{S}}_{\mathrm{ExhDA+\sigma A}} \forall v[(a \vee b)_v \rightarrow W_v]$

a. $\forall v[(a \vee b)_v \rightarrow W_v] \wedge \exists v[(a \vee b)_v] \wedge$

b. $\underbrace{\neg \overbrace{\underbrace{\mathrm{O}(\forall v[a_v \rightarrow W_v] \wedge \exists v[a_v])}_{(\forall v[a_v \rightarrow W_v] \wedge \exists v[a_v]) \wedge \neg(\forall v[b_v \rightarrow W_v] \wedge \exists v[b_v])}}}_{(\forall v[a_v \rightarrow W_v] \wedge \exists v[a_v]) \rightarrow (\forall v[b_v \rightarrow W_v] \wedge \exists v[b_v])} \wedge \underbrace{\neg \underbrace{\mathrm{O}(\forall v[b_v \rightarrow W_v] \wedge \exists v[b_v])}_{(\forall v[b_v \rightarrow W_v] \wedge \exists v[b_v]) \wedge \neg(\forall v[a_v \rightarrow W_v] \wedge \exists v[a_v])}}_{(\forall v[b_v \rightarrow W_v] \wedge \exists v[b_v]) \rightarrow (\forall v[a_v \rightarrow W_v] \wedge \exists v[a_v])}$

c. $\neg(\forall v[(a \wedge b) \rightarrow W_v] \wedge \exists v[(a \wedge b)_v \rightarrow W_v]$

(M1) (a) $\wedge \exists v[a_v] \wedge \exists v[b_v]$ (cf. $\mathrm{O}_{\mathrm{ExhDA+\sigma A}}(\Diamond(a \vee b)))$
'... and there is a world where Jo called Alice / individual who called Alice and there is a world where Jo called Bob / individual who called Bob.'

(M2) (a) $\wedge \neg\Box\exists v[a_v] \wedge \neg\Box\exists v[b_v]$ (cf. $\mathrm{O}_{\mathrm{ExhDA+\sigma A}}(\Box_{\mathrm{S}}(a \vee b)))$
'... and the speaker is ignorant/indifferent whether there is a world where Jo called Alice / individual who called Alice and ignorant/indifferent whether there is a world where Jo called Bob / individual who called Bob.'

The implications arising from the negations of the ExhDA can be true together *iff* both of their terms are true or both false. The $\forall$ components of each term are already entailed by the prejacent, so they cannot be false, so the truth of the implications depends only on the $\exists$ components. If both are true, that leads to strengthening as in (M1). If both are false, that can lead to strengthening via $\Box_{\mathrm{S}}$, as in (M2). Either way, what we see is that, once we factor in presuppositions, exhaustification of *or*/*some* NP_{SG} across *if*/*every* does in fact lead to strengthening.

Suppose now that some items reject a use of their alternatives that does not lead to strengthening while others don't. Chierchia (2013) calls this the Proper Strengthening parameter and conceptualizes this as a lexical presupposition on the use of the ExhDA.[4] For our $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}$, this can be spelled out as below.

(46) Exhaustification with Proper Strengthening:
$[\![\mathrm{O}^{\mathrm{PS}}_{\mathrm{ExhDA+}\sigma\mathrm{A}}(\mathrm{p})]\!]^{g,w}$ is defined iff $\lambda w.\, [\![\mathrm{O}^{\mathrm{S}}_{\mathrm{ExhDA}}(\mathrm{p})]\!]^{g,w} \subset p$.
Whenever defined, $[\![\mathrm{O}^{\mathrm{PS}}_{\mathrm{ExhDA+}\sigma\mathrm{A}}(\mathrm{p})]\!]^{g,w} = [\![\mathrm{O}^{\mathrm{S}}_{\mathrm{ExhDA+}\sigma\mathrm{A}}(\mathrm{p})]\!]^{g,w}$.

If both *or* and *some* NP_{SG} undergo exhaustification in the strong sense, but *some* NP_{SG} additionally carries a Proper Strengthening requirement, given that this requirement cannot be satisfied if these items are in the scope of negation but it can be satisfied if they are in the antecedent/restriction of *if/every*, this captures their distribution in (5)-(7).

Before we conclude, one last comment: If $\mathrm{O}^{\mathrm{S}}_{\mathrm{ExhDA}}$ can take into account non-truth-conditional content such as presuppositions, we may wonder if it can take into account implicatures also. For example, *few* below is a downward-entailing operator but it gives rise to a positive implicature. If $\mathrm{O}^{\mathrm{S}}_{\mathrm{ExhDA}}$took into account this implicature, this should lead to proper strengthenng, and *some* NP_{SG} should be felicitous. However, it is not, suggesting that, at least for *some* NP_{SG}, $\mathrm{O}^{\mathrm{S}}_{\mathrm{ExhDA}}$ can remain defined as above, that is, it only takes into account presuppositions.

(47) Few people believe that Jo called ✓Alice or Bob / # some student.

2.5 Comparison to previous literature

To my knowledge, none of the existing literature covers all of our starting patterns for *or*/*some* NP_{SG}, so in that sense the proposal above already represents empirical progress.

At the same time, there are other unified solutions to ignorance and polarity sensitivity, at least for disjunction, in the recent literature (e.g., Meyer 2013, Spector 2014, 2015, Nicolae 2017). Is our choice of a solution merely a matter of theoretical preference, or are there empirical advantages as well?

First, we used Chierchia (2013)'s contradiction-based exhaustivity operator O (so called because it can generate contradictions), but another option in the literature is to use Fox (2007)/Chierchia et al. (2012)'s contradiction-free / Innocent Exclusion-based definition of the exhaustivity operator, let's call it O_{IE} (it only negates those of the non-entailed alternatives that can be negated together while the prejacent remains

4 Note that here, but also in other cases, the σA-implicatures can in fact lead to local proper strengthening, e.g., if $\mathrm{O}_{\sigma\mathrm{A}}$ is computed below the downward-entailing operator, so this constraint can't depend on them.

true). A reason to stick with our O over $\mathrm{O_{IE}}$ is that it can be used to derive not just positive polarity but also negative polarity, and so can provide a unified solution to polarity sensitivity more generally (Chierchia 2013).

Second, we used ExhDA instead of plain DA. A reason to stick with ExhDA is that $\mathrm{O_{DA}}\Diamond(a\vee b) = \Diamond(a\vee b)\wedge\neg\Diamond a\wedge\neg\Diamond b, = \bot$ would yield only contradiction instead of Free Choice, and $\mathrm{O_{DA}}\Box_{\mathrm{S}}(a\vee b) = \Box_{\mathrm{S}}(a\vee b)\wedge\neg\Box_{\mathrm{S}}a\wedge\neg\Box_{\mathrm{S}}b$ would yield only ignorance, such that pruning of the σA-implicature wouldn't suffice to ensure compatibility with a no ignorance continuation. Without ExhDA, these results can be obtained only via recursive exhaustification with $\mathrm{O_{IE\text{-}DA}}$ (cf., e.g., Fox 2007 for the former and Nicolae 2017 for the latter), and these choices have far-reaching consequences in that we would have to cancel our assumption that an alternative feature can only be checked off once and we would no longer be able to use ExhDA vs. DA as a possible parameter of lexical difference between various items, thus losing some advantage in handling intervention effects or easily capturing differences in the distribution of certain lexical items (Chierchia 2013).

Third, we conceptualized the null modal that helped us capture ignorance as a last resort null modal $\Box$ with flavors potentially other than epistemic also. However, a popular alternative option has been to use Meyer (2013)'s K operator (a null matrix-level necessity modal like $\Box_{\mathrm{S}}$, but always available by default and always epistemic). A reason to stick with our $\Box$ over K is that it generates fewer logical forms and can capture more attested empirical variation (cf. Chierchia 2013 or Fălăuş 2014).

But, while all the previous advantages seemed to do mostly with external considerations, our choice of a recipe for ignorance and anti-negativity on which they go back to different factors — ignorance arises because $\mathrm{O_{ExhDA}}(a\vee b)$ without $\Box_{\mathrm{S}}$ yields contradiction, and anti-negativity because $\mathrm{O_{ExhDA}}(\neg(a\vee b))$ doesn't lead to proper strengthening and some items don't tolerate that — over the view that they go back to the same factor — $\mathrm{O_{IE\text{-}DA}}(a\vee b)$ and $\mathrm{O_{IE\text{-}DA}}(\neg(a\vee b))$ are both vacuous, and there is a ban against that, leading to a K fix for the former, capturing ignorance, and ungrammaticality for the latter, capturing anti-negativity — is simply because the latter view wouldn't capture our data: While it would capture *some* NP_{SG} and the French PPI disjunctions for which it was designed, it wouldn't capture the fact that, for example, *or* has a strong ignorance effect but no anti-negativity.

So far we have discussed differences with an alternative view of ignorance and polarity sensitivity different than Chierchia (2013)'s view that we adopted. However, some of our choices regarding pre-exhaustification and the solution to variation effects are different from Chierchia's also. That is primarily because Chierchia focused on variation effects with possibility modals, something that we didn't discuss at all, and the 'no winner' but not the 'one winner' case with necessity modals, thus leaving out one of our main patterns of interest. We will of course not be able to tackle all these issues properly here, but at a cursory glance the following differences emerge: In Chierchia's approach (a) the O used to generate ExhDA is actually $O_{\text{IE-DA}}$ (the idea being that pre-exhaustification, whose only goal is to strengthen the DA, must by design avoid contradiction); (b) pre-exhaustification of NonSgDA in general is done relative to both NonSgDA and SgDA; (c) pre-exhaustification of SgDA in all the DA cases is done relative to both SgDA and NonSgDA; and (d) the fact that in the configuration $O_{\text{ExhDA}}(\Diamond(a \vee b \vee c))$ some items are compatible only with a 'no winner' interpretation ($\Diamond a \wedge \Diamond b \wedge \Diamond c$) as opposed to a 'one loser' interpretation ($\neg\Diamond a \wedge \Diamond b \wedge \Diamond c$) is derived as arising from exhaustification relative to just ExhNonSgDA vs. just ExhSgDA. We didn't need either (a) or (b) in our data, yet they are crucial for embeddings under possibility modals; we accept them as necessary refinements and note that they don't seem to affect in any way our results for the necessity modal cases so far. However, (c) would have a bad consequence for our computations: The SgDA-implicatures become compatible with the positive certainty / 'one winner' case, so the overall result can no longer be total ignorance, contrary to what we want. In conjunction with (b), we conclude that the SgDA must be pre-exhaustified relative to just other SgDA, but the NonSgDA must be pre-exhaustified relative to a set consisting of other NonSgDA and their own smaller DA. Finally, regarding (d), the same items that have only the 'no winner' interpretation under possibility modals are in fact items that are incompatible with the 'one winner' case under necessity modals — the case we obtained from ExhNonSgDA. We conclude that the contrast in (d) is in fact not a contrast between having to exhaustify relative to just ExhNonSgDA vs. ExhSgDA, but rather a contrast between having to exhaustify relative to all the DA vs. being able to exhaustify relative to just the SgDA. All these remarks are, of course, very preliminary, but we thought them useful to facilitate future discussion.

2.6 Summary

In this section we used Chierchia (2013)'s approach to similarity and variation with respect to ignorance and polarity to articulate an account for *or/some* NP_{SG} and extract a general recipe for how to handle these phenomena more generally. In the next section we will use this recipe to develop an account of ignorance and polarity sensitivity in CMNs/SMNs.

3 CMNs and SMNs

There are many approaches to ignorance and polarity sensitivity in CMNs/SMNs. However, none captures all the patterns. And while many approaches draw some parallelism with disjunction, this is usually done only for SMNs, the source of the parallelism is typically identified only in the metalanguage — e.g., *at least three* is paraphrased as 'exactly three or more than three/at least four', inspired from the disjunctive nature of the ordering relation traditionally used in its truth conditions ($\geq$ meaning $>$ or $=$) — and the parallelism itself is formally confined to the use of alternatives based on the disjuncts — e.g., the alternatives of *at least three* are assumed to be *exactly three* and *more than three / at least four*. In our search for an account of ignorance and polarity sensitivity in CMNs/SMNs that would not only explain all our starting patterns but moreover do so in a way that captures their similarity to *or/some* NP_{SG}, it is thus better if we start from scratch, retracing our own steps for *or/some* NP_{SG}. As we will see, with certain new but reasonable assumptions about the truth conditions of CMNs and SMNs, the approach to ignorance and polarity sensitivity that we outlined for *or/some* NP_{SG} can be seamlessly extended to capture our starting patterns for CMNs/SMNs also, and moreover sheds new light on a theory of bare and modified numerals more generally. At the end we compare the resulting theory of CMNs/SMNs to existing accounts.

3.1 Truth conditions and alternatives

The first crucial step in our solution for ignorance and polarity sensitivity in *or/some* NP_{SG} was to spell out their truth conditions and alternatives. However, there is no consensus about the truth conditions or alternatives of CMNs/SMNs. In fact there is also no consensus about the truth conditions or alternatives of bare numerals (henceforth, BNs) either. In what follows we aim to give a principled answer for all.

Let's start with the truth conditions.

First, consider a simple BN utterance, e.g., *Three people quit*. I adopt the view that *three* denotes a simple degree, type *d* (Buccola & Spector 2016).[5] It composes with the meaning of *people* via the intermediary of a cardinality operator [count], the result of the composition of [count](people)(three) being a predicate that when applied to an individual yields true *iff* it is a plurality with atom count 3 of people (cf. Zabbal 2005, Scontras 2013 and references therein). A silent determiner with the meaning of an existential quantifier (Link 1987) takes as an argument this predicate and the predicate that is the meaning of *quit* and yields true *iff* there exists a plurality with atom count 3 of people who quit. All these are spelled out in (48). A derivation tree is given in Figure 1. (The syntactic assumptions about [count] being the head of a functional projection Num(ber)P intermediary between the DP and the NP and the numeral being a phrasal projection merged in the specifier of NumP are as in Zabbal 2005, Scontras 2013 and references therein.) Modulo the details of the composition,

5 To capture the predicative use of *three* in *We are three*, I assume with Buccola & Spector (2016) that this degree meaning can be typeshifted into a predicative meaning via a typeshifter $[\![\text{isCard}]\!] = \lambda n\,.\,\lambda x\,.\,|x| = n$. Typeshifting in the opposite direction, that is, from a predicative to a degree meaning, is however also conceivable.

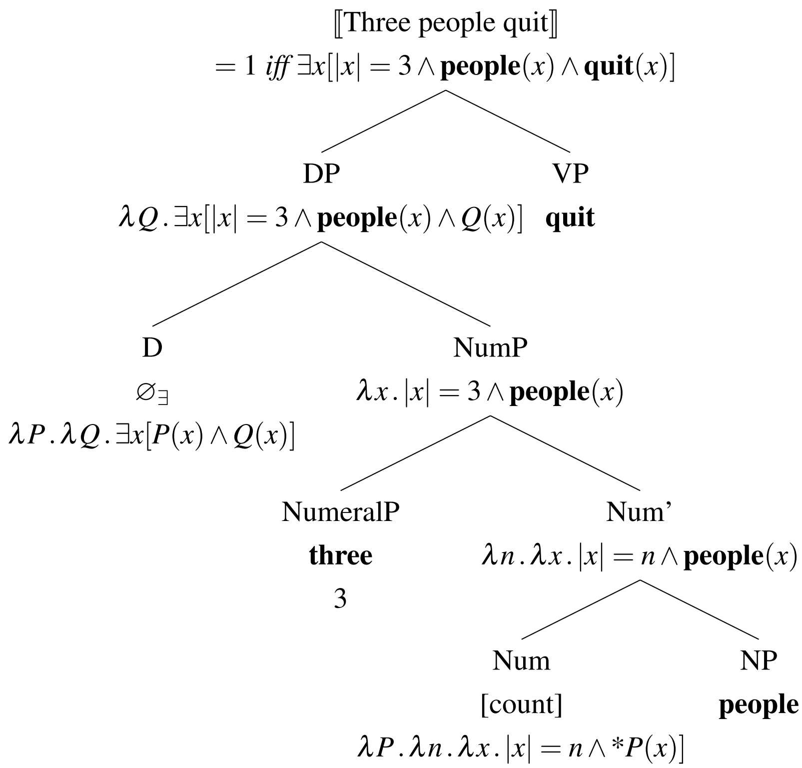

Figure 1 The syntax and semantics of BNs.

these are the so-called existential truth conditions of BNs from Link (1987), Krifka (1999), or Kennedy (2015).

(48) Three people quit.
$[\![(\varnothing_\exists$ (three ([count] people))) (quit)$]\!]$
a. $[\![$three$]\!] = 3$ (type d)
b. $[\![$[count]$]\!] = \lambda P \,.\, \lambda n \,.\, \lambda x \,.\, |x| = n \wedge {}^*P(x)]$
c. $[\![$[count] people$]\!] = \lambda n \,.\, \lambda x \,.\, |x| = n \wedge \mathbf{people}(x)$
d. $[\![$three ([count] people)$]\!] = \lambda x \,.\, |x| = 3 \wedge \mathbf{people}(x)$
$= 1$ *iff* $\exists x[|x| = 3 \wedge \mathbf{people}(x) \wedge \mathbf{quit}(x)]$

Now, let's consider a simple CMN/SMN utterance, e.g., *More/less than three people quit / At least/most people quit*. I adopt the view that *more/less than three / at least/most three* denote generalized quantifiers over degrees, type $\langle dt,t\rangle$ (Hackl 2000, Kennedy 2015). As such, they are unable to compose with the meaning of [count](P) (see above for BNs), which is type $\langle d,et\rangle$, in situ, and so must move. At the origin site this movement creates a trace type d that composes with [count](P) the way a BN would. At the destination site this movement gives rise to a degree predicate that goes on to saturate the degree predicate argument of the modified numeral. Modulo the details of the composition, so far this resembles the treatment of CMNs/CMNs and SMNs from Hackl (2000)/Kennedy (2015).

The steps that follow are new, and have to do with a decomposition of the modifiers. Note that *more than/at most / less than/at least* share a *much/little* meaning; *more/less than* share a comparative meaning, let's call it [comp]; and *at least/most* share an *at*-superlative meaning, let's call it [at-sup]. How do the meanings of CMNs/SMNs come together from these pieces?

First, I propose that the *much/little* in the meaning of CMNs/SMNs are positive/negative extent indicators (idea adapted from Kennedy 2001, Seuren 1984's extent view of adjectives). More concretely, *much/little* are functions type $\langle d, dt \rangle$, defined as follows. *Much* maps a degree n to the set of degrees on a relevant scale that are less than or equal to n's measure on the scale, and *little* maps a degree n to the set of degrees on a relevant scale that are greater than or equal to n's measure on the scale. Assuming the measure of a degree n on any scale is simply n, these meanings come down to the following:

(49) $[\![\text{much}]\!] = \lambda n \,.\, \lambda d \,.\, d \leq n$

e.g., $[\![\text{much}]\!](3) = \lambda d \,.\, d \leq 3$

(50) $[\![\text{little}]\!] = \lambda n \,.\, \lambda d \,.\, d \geq n$

e.g., $[\![\text{little}]\!](3) = \lambda d \,.\, d \geq 3$

Second, I propose that [comp]/[at-sup] are functions type $\langle\langle d, dt \rangle, \langle d, \langle dt, t \rangle\rangle\rangle$ which take as an argument *much/little*, a numeral n, and a degree predicate D. Then, $[\![\text{[comp](much/little)(n)(D)}]\!]$ yields true *iff* the maximum of D is in $\overline{[\![\text{much/little}]\!](n)}$ — that is, in the complement of the positive/negative extent of n. And $[\![\text{[at-sup](much/little)(n)(D)}]\!]$ yields true *iff* the maximum of D is in $[\![\text{much/little}]\!](n)$ — that is, in the positive/negative extent of n. All these are spelled out in (51)-(52). A derivation tree is given in Figure 2.

(51) More/less than three people quit.
$[\![(([\text{comp}]\ (\text{much/little}))\ (\text{three}))\ (1\ \varnothing_{\exists}\ t_1\ \text{people quit})]\!]$
a. $[\![[\text{comp}]]\!] = \lambda f_{\langle d,dt \rangle} \,.\, \lambda n_d \,.\, \lambda D_{\langle d,t \rangle} \,.\, \max(\lambda d \,.\, D(d)) \in \overline{f(n)}$
$= 1 \text{ iff } \max(\lambda d \,.\, \exists x[|x| = d \wedge P(x) \wedge Q(x)]) \in \underbrace{\overline{[\![\text{much/little}]\!](3)}}_{\{4,5,...\}/\{0,1,2\}}$

(52) At most/least three people quit.
$[\![(([\text{at-sup}]\ (\text{much/little}))\ (\text{three}))\ (1\ \varnothing_{\exists}\ t_1\ \text{people quit})]\!]$
a. $[\![[\text{at-sup}]]\!] = \lambda f_{\langle d,dt \rangle} \,.\, \lambda n_d \,.\, \lambda D_{\langle d,t \rangle} \,.\, \max(\lambda d \,.\, D(d)) \in f(n)$

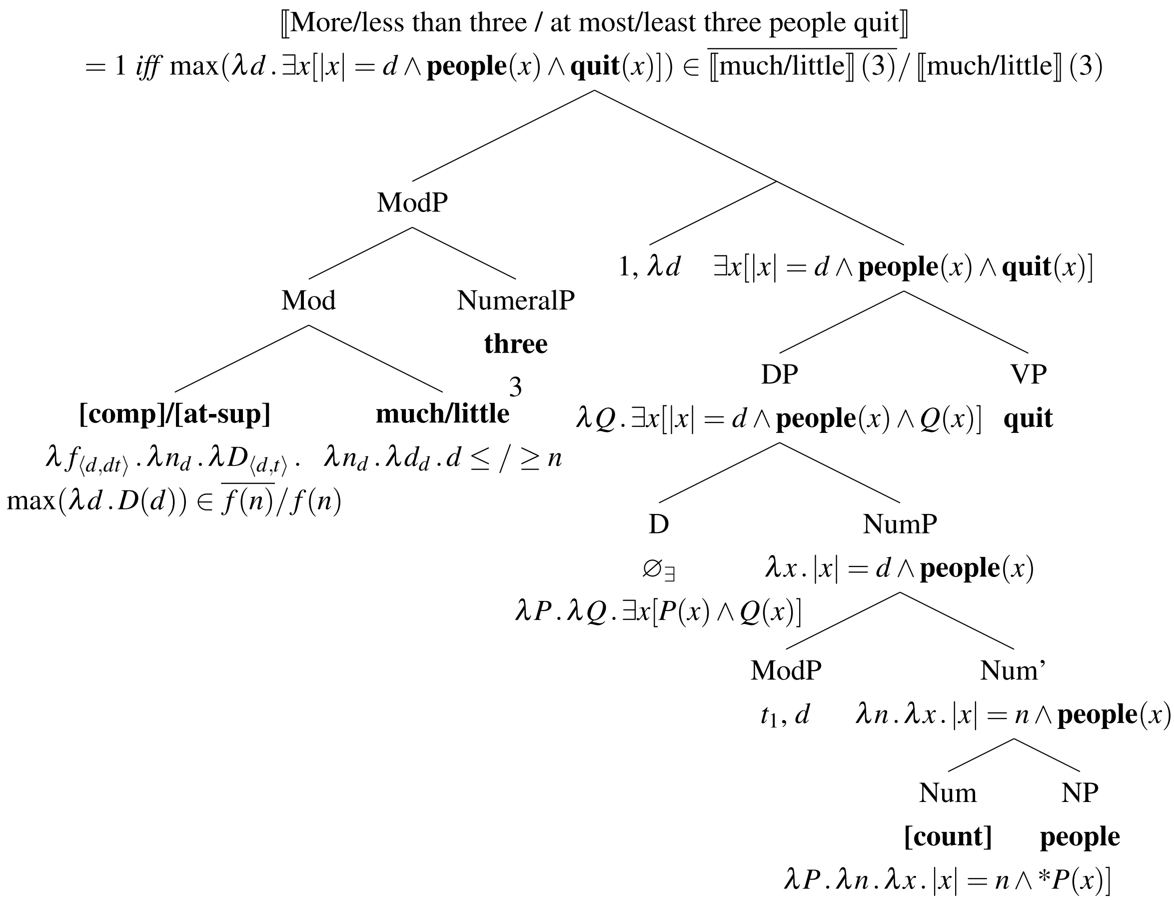

Figure 2 The syntax and semantics of CMNs and SMNs.

$$= 1 \text{ iff } \max(\lambda d\,.\,\exists x[|x| = d \wedge P(x) \wedge Q(x)]) \in \underbrace{[\![\text{much/little}]\!]\,(3)}_{\{0,1,2,3\}/\{3,4,5,\ldots\}}$$

Although they offer a more detailed decomposition of the modifiers, these truth conditions are fundamentally similar to existing proposals in the literature. Where they give us a vital advantage is in the generation of alternatives. In the existing proposals the truth conditions of all of BNs, CMNs, and SMNs are given in terms of just a scalar item, the numeral ($= 3$, $> / < 3$, $\geq / \leq 3$). Thus, if we confine ourselves to our traditional alternative generation mechanism, we get only scalar alternatives. However, in our proposal the truth conditions of CMNs and SMNs also make reference to a natural domain — the set of degrees based around the numeral that gives us the value of max. Thus, even if we confine ourselves to just our traditional alternative generation mechanism, we are able to straightforwardly get both subdomain and scalar alternatives.[6] These are listed below.

(53) n people quit.

a. $\exists x[|x| = n \wedge P(x) \wedge Q(x)]$ (assertion)
b. – (no DA)
c. $\{\exists x[|x| = m \wedge P(x) \wedge Q(x)] \mid m \in S\}$ (σA)

(54) More/less than n people quit.

a. $\max(\lambda d\,.\,\exists x[|x| = d \wedge P(x) \wedge Q(x)]) \in \overline{\llbracket \text{much/little} \rrbracket\,(n)}$ (assertion)
b. $\{\max(\lambda d\,.\,\exists x[|x| = d \wedge P(x) \wedge Q(x)]) \in D' \mid D' \subset \overline{\llbracket \text{much/little} \rrbracket\,(n)}\}$ (DA)
c. $\{\max(\lambda d\,.\,\exists x[|x| = d \wedge P(x) \wedge Q(x)]) \in \overline{\llbracket \text{much/little} \rrbracket\,(m)} \mid m \in S\}$ (σA)

(55) At most/least n people quit.

a. $\max(\lambda d\,.\,\exists x[|x| = d \wedge P(x) \wedge Q(x)]) \in \llbracket \text{much/little} \rrbracket\,(n)$ (assertion)
b. $\{\max(\lambda d\,.\,\exists x[|x| = d \wedge P(x) \wedge Q(x)]) \in D' \mid D' \subset \llbracket \text{much/little} \rrbracket\,(n)\}$ (DA)
c. $\{\max(\lambda d\,.\,\exists x[|x| = d \wedge P(x) \wedge Q(x)]) \in \llbracket \text{much/little} \rrbracket\,(m) \mid m \in S\}$ (σA)

For example, *three* has the truth conditions and alternatives below, henceforth abbreviated as on the right:

(56) Three people quit.

a. $\exists x[|x| = 3 \wedge P(x) \wedge Q(x)]$ (assertion; abbr. $3 \vee 4 \vee \ldots$)
b. — (no DA)
c. $\ldots$ $\ldots$
$\exists x[|x| = 2 \wedge P(x) \wedge Q(x)]$ (σA; abbr. $2 \vee 3 \vee \ldots$)
$\exists x[|x| = 4 \wedge P(x) \wedge Q(x)]$ (σA; abbr. $4 \vee 5 \vee \ldots$)
$\ldots$ $\ldots$

And, *less than two* / *at most one* — a CMN/SMN pair with the same two-element domain — have the truth conditions and alternatives below (where max stands for $\max(\lambda d\,.\,\exists x[|x| = d \wedge P(x) \wedge Q(x)])$), henceforth abbreviated as shown on the right:

(57) Less than two people quit. / At most one person quit.

a. $\max \in \underbrace{\overline{\llbracket \text{little} \rrbracket\,(2)} / \llbracket \text{much} \rrbracket\,(1)}_{\{0,1\}}$ (assertion; abbr. $0 \vee 1$)
b. $\max \in \{0\}$ (singleton DA; abbr. 0)
$\max \in \{1\}$ (singleton DA; abbr. 1)

6 Note that there are no new mixed-type alternatives here — not all successful replacements of the domain with its subset correspond to a successful replacement of the scalar element, and those that do are already identical to existing σA, so they are not generated.

c. $\max \in \underbrace{\overline{[\![\text{little}]\!](1)}/[\![\text{much}]\!](0)}_{\{0\}}$ (σA; abbr. 0)

$\max \in \underbrace{\overline{[\![\text{little}]\!](3)}/[\![\text{much}]\!](2)}_{\{0,1,2\}}$ (σA; abbr. $0 \vee 1 \vee 2$)

Finally, *less than two* / *at most one* — a CMN/SMN pair with the same three-element domain — have the truth conditions and alternatives below (where max stands for $\max(\lambda d\,.\,\exists x[|x| = d \wedge P(x) \wedge Q(x)])$), henceforth abbreviated as shown on the right:

(58) Less than three people quit. / At most two people quit.

a. $\max \in \underbrace{\overline{[\![\text{little}]\!](3)}/[\![\text{much}]\!](2)}_{\{0,1,2\}}$ (assertion; abbr. $0 \vee 1 \vee 2$)

b. $\max \in \{0\}$ (singleton DA; abbr. 0)
$\max \in \{1\}$ (singleton DA; abbr. 1)
$\max \in \{2\}$ (singleton DA; abbr. 2)
$\max \in \{0,1\}$ (doubleton DA; abbr. $0 \vee 1$)
$\max \in \{0,2\}$ (doubleton DA; abbr. $0 \vee 2$)
$\max \in \{1,2\}$ (doubleton DA; abbr. $1 \vee 2$)

c. $\max \in \underbrace{\overline{[\![\text{little}]\!](1)}/[\![\text{much}]\!](0)}_{\{0\}}$ (σA; abbr. 0)

$\max \in \underbrace{\overline{[\![\text{little}]\!](2)}/[\![\text{much}]\!](1)}_{\{0,1\}}$ (σA; abbr. $0 \vee 1$)

$\max \in \underbrace{\overline{[\![\text{little}]\!](4)}/[\![\text{much}]\!](3)}_{\{0,1,2,3\}}$ (σA; abbr. $0 \vee 1 \vee 2 \vee 3$)

... ...

3.2 Exhaustification

As for *or*/*some* NP_{SG}, we will assume that CMNs/SMNs are exhaustified via O, and that both their ExhDA and their σA are factored in by default. As for *or*/*some* NP_{SG}, in the simplest form this means that they have to be in the scope of $\mathrm{O}_{\text{ExhDA}+\sigma\text{A}}$.

3.3 Scalar implicatures

On our meanings for BNs, CMNs, and SMNs their truth conditions are like those of *or*/*some* NP_{SG}, that is, single-bounded — *three*, *more than three*, and *at least three* all entail just a lower bound, and *less than three*, *at most three* all entail just an upper bound. At the same time, just like *or*/*some* NP_{SG}, they activate scalar alternatives and so should implicate a second bound. That is, just like an utterance of *Jo called Alice or Bob* σA-implicated that she didn't call both, an utterance of *Jo called three / more than three / at least three people* should σA-implicate that she didn't call four / more than four / at least four, and an utterance of *Jo called less than three / at most three people* should σA-implicate that she didn't call less than two / at most two. And, as regular σA-implicatures, these second bounds should be cancelable, e.g., via an *in fact* continuation.

Now, although, in light of everything we know about *or*/*some* NP_{SG} so far, this view seems sound, it has been challenged for both BNs and CMNs/SMNs. Before we continue with our discussion of ignorance and polarity sensitivity, we must thus first engage with the challenges and explain why we want to maintain this position.

Let's first begin with BNs. In a plain episodic context the most salient interpretation for a BN is a double-bounded one. That is, *three* doesn't just carry a negative inference about values below three, but also a negative inference about values above three — altogether, it is interpreted as 'exactly three'. However, while the negative inference about the values below three cannot be canceled and is therefore an entailment, the negative inference about the values above three can be canceled.

(59) a. Jo called three people.
b. (i) $\neg$ The number of people that Jo called is two or less.
(ii) $\neg$ The number of people that Jo called is four or more.
c. Jo called three people, # if not less / ✓if not more.

This is precisely consistent with our view of *three*, on which the lower bound is an entailment coming from its truth conditions whereas the upper bound is an implicature coming from its σA.

(60) Jo called three people.
$O_{\sigma A}(3 \vee 4 \vee \ldots)])$
a. $(3 \vee 4 \vee \ldots) \wedge$ (prejacent)
b. $\neg(4 \vee \ldots)$ (σA-implicature)
$= (3 \vee 4 \vee \ldots) \wedge \neg(4 \vee \ldots),\ = 3$
'Jo called exactly three people.' ✓

At the same time, it has been pointed out that the strengthened ('exactly') meaning of BNs contrasts with the strengthened meaning of other scalar items in being a lot more natural and available in a variety of settings (cf., e.g., Horn 1992, 1996, Kennedy 2013 for introspective judgments and Noveck 2001, Papafragou & Musolino 2003, Musolino 2004, Guasti et al. 2005, Pouscoulous et al. 2007, Huang & Snedeker 2009, Marty et al. 2013, Huang et al. 2013 for experimental work). As a result of this research, the optionality of the σA-implicature above has been reinterpreted as evidence of a lexical ambiguity of BNs between an 'at least' and an 'exactly' meaning (in different forms, e.g., Geurts 2006, Kennedy 2015).

Turning now to CMNs and SMNs, it has been pointed out that, if they gave rise to σA-implicatures the way we just showed for BNs, they should all give rise to 'exactly' meanings also, for the same exact reason as in the case of BNs. However, this is obviously not the way we interpret them as a default.

(61) Jo called less than three / at most three people.
$O_{\sigma A}(0 \vee 1 \vee 2)])$
a. $(0 \vee 1 \vee 2) \wedge$ (prejacent)
b. $\neg(0 \vee 1)$ (σA-implicature)
$= (0 \vee 1 \vee 2) \wedge \neg(0 \vee 1), = 2$
'Jo called exactly two people.' ✗

There have been a wide variety of responses to this. Some proposals tweak the modifiers (Krifka 1999, Geurts & Nouwen 2007, Coppock & Brochhagen 2013), others the scale (Fox & Hackl 2006), and yet others the set of alternatives themselves, in many different ways (Büring 2008, Mayr 2013, Kennedy 2015, Spector 2015, Schwarz 2016). What they all have in common, however, is a departure from the idea that CMNs/SMNs give rise to classic σA-implicatures.

Should we follow this literature and abandon the idea that BNs and CMNs/SMNs give rise to σA-implicatures? Before we do that, let's review a few more patterns.

First, the fact that σA-strengthening is more automatic and available for BNs than for other scalar items could be explained by saying that for BNs the scale is more automatic / better known / better defined for BNs, and/or that BNs are intrinsically focused (Spector 2013), so essentially they must always as a default undergo $O_{\sigma A}$ / are less affected by constraints on uses of $O_{\sigma A}$ that lead to weakening (e.g., as in the case where $O_{\sigma A}$ is computed below a negative operator).

Second, note that both BNs and CMNs/SMNs can all in fact give rise to the expected σA-implicatures if these are computed across an overt universal operator. I illustrate for *three* / *more than two* / *at least three*.

(62) Jo is required to call three / more than three / at least three people.
$O_{\sigma A}\Box(3 \vee 4 \vee \ldots)$
a. $\Box(3 \vee 4 \vee \ldots) \wedge$ (assertion)
b. $\neg\Box(4 \vee \ldots)$ (σA-implicature)
'…but she is not required to call four / more than four / at least four.'

Also, both BNs and CMNs/SMNs can give rise to the expected σA-implicatures when these implicatures are computed across a downward-entailing operator such as *if/every*.[7] I illustrate below for *three* / *more than two* / *at least three*.

(63) If Jo called three / more than three / at least three people, she won.
$O_{\sigma A} \forall v((3 \vee 4 \vee \ldots)_v \to W_v)$
a. $\forall v((3 \vee 4 \vee \ldots)_v \to W_v) \wedge$ (assertion)
b. $\neg \forall v((2 \vee 3 \vee 4 \vee \ldots)_v \to W_v)$ (σA-implicature)
$\rightsquigarrow$ '…but not if she called two / more than two / at least two.'

While most of the existing theories of BNs, CMNs, and SMNs have a way to handle (62) (a recent example is Kennedy 2015), none have a way to capture (63). On the other hand, as we just saw, the classical view captures both with equal ease.

Third, if the fact that an σA-implicature overgenerated an illicit 'exactly' meaning like the one we saw for CMNs/SMNs were a sufficient reason to abandon the classic view of σA-implicatures for CMNs/SMNs, then BNs are in fact in trouble also, as $O_{\sigma A}$ across negation leads to an illicit 'exactly' meaning for all.

(64) Jo didn't call three / more than two / # at least three people.
$O_{\sigma A} \neg(3 \vee 4 \vee \ldots)$
a. $\neg(3 \vee 4 \vee \ldots)$ (assertion)
b. $\neg\neg(2 \vee 3 \vee 4 \vee \ldots)$ (σA-implicature)
$= \neg(3 \vee 4 \vee \ldots) \wedge (2 \vee 3 \vee 4 \vee \ldots),\ = 2$
'Jo called exactly two people.' ✗

Thus, a better response to these illicit results in CMNs/SMNs is not to abandon the σA-implicatures but rather to look for a way to rule them out.

Fourth, all the σA-implicatures that led to undesirable 'exactly' meanings actually lead to sensible and desirable meanings if we simply prune the set of σA (cf. also example in Spector 2014: 41 for CMNs/SMNs in the positive context).

(65) Jo called less than three / # at most three people.
$O_{\sigma A}(0 \vee 1 \vee 2)])$
a. $(0 \vee 1 \vee 2) \wedge$ (prejacent)
b. $\neg 0$ (σA-implicature after pruning the σA $(0 \vee 1)$)
$= (0 \vee 1 \vee 2) \wedge \neg 0,\ = 1 \vee 2$
'Jo called less than three / at most three people, but she did call some.' ✓

7 These are sometimes called *indirect* σA-implicatures, cf. Chierchia (2004), reflecting the fact that they are based on σA that are normally weaker than the assertion but which the addition of the downward-entailing operator has made stronger by reversing the scale. They are also not limited to $O_{\sigma A}$across *if/every* — Spector (2013: 279-80), e.g., shows the same for $O_{\sigma A}$ across *not allowed*.

(66) Jo didn't call three / more than two / # at least three people.
$O_{\sigma A}\neg(3\vee 4\vee\ldots)$
a. $\neg(3\vee 4\vee\ldots)$ (assertion)
b. $\neg\neg(1\vee 2\vee\ldots)$ (σA-implicature after pruning the σA $(2\vee 3\vee\ldots)$)
$=\neg(3\vee 4\vee\ldots)\wedge(1\vee 2\vee 3\vee\ldots),=1\vee 2$
'Jo didn't call three / more than two / # at least three people, but she did call some.' ✗

This reinforces the idea that the problem is not with the traditional σA-implicatures as a whole, but only with certain illicit 'exactly' meanings.

Last but not least, not only are there many reasons based on introspective data like the above to maintain the original σA-implicatures, but there is also experimental research to support it (cf., e.g., Barner & Bachrach 2010, Panizza et al. 2009 for experimental evidence that the strengthened meaning of BNs is sensitive to knowledge of scale / the monotonicity of the embedding environment, like that of a regular scalar; and Cummins et al. (2012) for σA-implicatures in CMNs/SMNs).

Given all of these, we will conclude that BNs, CMNs, and SMNs do in fact give rise to σA-implicatures as on the classical view. We will provide a way to rule out the illicit 'exactly' meanings for CMNs and SMNs in the positive case at the end of our discussion of ignorance in Section 3.4, and a way to rule out the illicit 'exactly' meanings of BNs, CMNs, and SMNs in the negative case at the end of our discussion of polarity sensitivity in Section 3.5.

3.4 Ignorance

As for *or*/*some* NP_{SG}, let's first consider exhaustification via $O_{\text{ExhDA}+\sigma A}$ without any intervening operator. As usual, $O_{\text{ExhDA}+\sigma A}$ assserts the prejacent, (67-a), negates the non-entailed ExhDA, (67-b), and negates the non-entailed σA, (67-c) (as for *or*/*some* NP_{SG}, below and going forward we will always list these on separate lines, for clarity; for a refresher on the alternatives, see (57)).

(67) Jo called less than two people / at most one person.
$O_{\text{ExhDA}+\sigma A}(0\vee 1)$
a. $(0\vee 1)\wedge$ (prejacent)
b. $\wedge\neg\underbrace{\underbrace{O0}_{0\wedge\neg 1}}_{0\rightarrow 1}\wedge\neg\underbrace{\underbrace{O1}_{1\wedge\neg 0}}_{1\rightarrow 0}\wedge$ (ExhDA-implicatures)
c. $\neg 0$ (σA-implicatures)
$=\bot$ (G-trivial)

Because of the different nature of the domain and the relation between the DA and the σA in *or/some NP$_{SG}$* vs. CMNs/SMNs, this computation goes a little bit differently: For *or/some NP$_{SG}$* the implications could be resolved to either false-false or true-true; however, here, because the nature of the domain which is such that no two values can be concomitantly true, the implications can only be resolved to false-false. Then, for *or/some NP$_{SG}$* the false-false option clashed with the prejacent, and the true-true option clashed with the σA-implicature. For CMNs/SMNs the unique false-false option clashes with the prejacent, and the σA-implicature makes no difference. Still, the result in both cases is the same: contradiction. As we will see, this difference in the details but equivalence in the result is typical of all the exhaustifications that follow.

As for *or/some NP$_{SG}$*, the result above makes us wonder: How can we capture our starting pattern below, which shows that an utterance as the above is grammatical and moreover gives rise to ignorance?

(9) (How many people did Jo call?) Jo called less than two people / at most one person. ($\rightsquigarrow$ ignorance)

And, as before, we will remark that ignorance feels like a silent modal effect, and in an attempt to get a better grasp on it we will proceed to consider cases where CMNs/SMNs are exhaustified via $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}$ across overt modals.

Consider first $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}$ across an overt possibility modal.

(68) Jo may call less than two people / at most one person.
$\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}(\Diamond(0 \vee 1))$
a. $\Diamond(0 \vee 1) \wedge$
b. $\underbrace{\neg \underbrace{\mathrm{O}(\Diamond 0)}_{\Diamond 0 \wedge \neg \Diamond 1}}_{\Diamond 0 \rightarrow \Diamond 1} \wedge \underbrace{\neg \underbrace{\mathrm{O}(\Diamond 1)}_{\Diamond 1 \wedge \neg \Diamond 0}}_{\Diamond 1 \rightarrow \Diamond 0} \wedge$
c. $\neg \Diamond 0$
$= \Diamond(0 \vee 1) \wedge \Diamond 0 \wedge \Diamond 1 \cancel{\wedge \neg \Diamond 0}$
'There is an accessible world where Jo calls 0 or 1 people and there is an accessible world where she calls 0, and there is an accessible world where she calls 1~~, and there is no accessible world where she calls 0~~.'

Here too there is a difference: While, as for *or/some NP$_{SG}$*, the ExhDA-implicatures again yield a Free Choice effect, differently from *or/some NP$_{SG}$*, in this case the σA-implicature ends up neutralizing this effect. This is an unexpected and undesirable result, and ideally we would want to avoid it by removing the σA-implicature. However, for *or/some NP$_{SG}$* we only allowed an σA-implicature to be suspended if forced by the context. Yet here we would want to eliminate it simply because it clashes with the ExhDA-implicature. Would that be inconsistent? Not necessarily: Note that the role of this σA-implicature was different for *or/some NP$_{SG}$* than it is now for CMNs/SMNs. There its general effect was to modulate the result of the ExhDA-implicatures such that *or/some NP$_{SG}$* didn't end up meaning the same as their formal σA *and/every*. However, here it doesn't achieve that — the result would be $\Diamond 1$, which is not one of the formal σA of $\Diamond$ *less than two/at most one* (see (57)). I will assume that, because of this difference in the effect of the σA-implicature in the two cases, the notion of σA-implicature cancellation can be different: For *or/some NP$_{SG}$*, the σA-implicatures are suspended if they clash with the context, but otherwise they are always kept, along with their consequences for the ExhDA-implicatures. However, for CMNs/SMNs, while their σA-implicatures are also suspended if they clash with the context, they can also be suspended/weakened via σA-pruning if they clash with the ExhDA-implicatures, as a default way to fix this clash. We will see more of this later on. The bottom line is, there are reasons to think that the result here is in fact precisely as for *or/some NP$_{SG}$*, that is, a Free Choice effect, just $\Diamond 0 \wedge \Diamond 1$.[8]

Consider now $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}$ across an overt necessity modal.

(69) Jo must call less than two people / at most one person.
$\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}(\Box(0 \vee 1))$

a. $\Box(0 \vee 1) \wedge$

b. $\neg \underbrace{\underbrace{\mathrm{O}(\Box 0)}_{\Box 0 \wedge \neg \Box 1}}_{\Box 0 \rightarrow \Box 1} \wedge \neg \underbrace{\underbrace{\mathrm{O}(\Box 1)}_{\Box 1 \wedge \neg \Box 0}}_{\Box 1 \rightarrow \Box 0} \wedge$

c. $\neg \Box 0$

$= \underbrace{\Box(0 \vee 1) \wedge \neg \Box 0 \wedge \neg \Box 1}_{\Box(0 \vee 1) \wedge \Diamond 0 \wedge \Diamond 1} \wedge \neg \Box 0$

'In every accessible world Jo calls 0 or 1 people, and it is not the case that

8 We used this parse primarily to see the Free Choice effect and as an early opportunity to comment on σA-implicature suspension in CMNs/SMNs as opposed to *or/some NP$_{SG}$*. However, note that other parses may be possible also. For example, another possible parse is a split exhaustification parse such as $\mathrm{O}_{\sigma\mathrm{A}}(\mathrm{O}_{\mathrm{ExhDA}}(\Diamond(0 \vee 1))$. The prejacent for $\mathrm{O}_{\sigma\mathrm{A}}$ here is $\mathrm{O}_{\mathrm{ExhDA}}(\Diamond(0 \vee 1) = \Diamond 0 \wedge \Diamond 1$, and the σA are $\{\mathrm{O}_{\mathrm{ExhDA}}(\Diamond 0) = \Diamond 0$, $\mathrm{O}_{\mathrm{ExhDA}}(\Diamond(0 \vee 1 \vee 2)) = \Diamond 0 \wedge \Diamond 1 \wedge \Diamond 2, \dots\}$. Note that, due to $\mathrm{O}_{\mathrm{ExhDA}}$, it is in fact the σA based on a larger numeral that are stronger and must be excluded. The result is $\Diamond 0 \wedge \Diamond 1 \wedge \neg \Diamond 2$. This parse in fact captures a known and tricky puzzle about *may ... at most n*, namely, that the fact that its most salient interpretation seems to be one where values higher than *n* are forbidden. However, it also predicts a similar upper bound for *may ... less than n* and a lower bound for *may ... more than/at least*, but these don't seem to match intuitions. We will not be able to discuss this, or possibility modal cases, any further going forward; we leave them to future research.

in every world she calls 0, and it is not the case that in every world she calls 1, and it is not the case that in every world she calls 0.'

The implications arising from the ExhDA are consistent only with $\neg\Box 0 \wedge \neg\Box 1$. The σA-implicature makes no difference. Together with the prejacent, the ExhDA-implicatures yield a Free Choice effect. The end result is as for *or*/*some* NP_{SG}.

At this point in our discussion of *or*/*some* NP_{SG} we said that what looked like an occurrence of these items in an episodic context was really an occurrence under a silent, matrix-level, speaker-oriented necessity modal. We will make the same assumption for CMNs/SMNs also.

(70) Jo called less than two people / at most one person.
$O_{\text{ExhDA}+\sigma\text{A}}(\Box_S(0 \vee 1))$
a. $\Box_S(0 \vee 1) \wedge$
b. $\underbrace{\neg \underbrace{O(\Box_S 0)}_{\Box_S 0 \wedge \neg\Box_S 1}}_{\Box_S 0 \rightarrow \Box_S 1} \wedge \underbrace{\neg \underbrace{O(\Box_S 1)}_{\Box_S 1 \wedge \neg\Box_S 0}}_{\Box_S 1 \rightarrow \Box_S 0} \wedge$
c. $\neg\Box_S 0$
$= \underbrace{\Box_S(0 \vee 1) \wedge \neg\Box_S 0 \wedge \neg\Box_S 1}_{\Box_S(0 \vee 1) \wedge \Diamond 0 \wedge \Diamond 1} \wedge \neg\Box_S 0$
'In every accessible world Jo called 0 or 1 people, and it is not the case that in every world she called 0, and it is not the case that in every world she called 1, and it is not the case that in every world she called 0.'

As for *or*/*some* NP_{SG}, note that the result is an epistemic Free Choice effect — ignorance. This captures our starting ignorance effect in (9). (As before, I will assume that this modal is inserted as a last resort rescue mechanism, and that it can have flavors such as agent-oriented bouletic also, being able to also yield indifference.)

At this point in our discussion of *or*/*some* NP_{SG} we noted that the ignorance effect obtained this way is total — the speaker is ignorant about every degree in the domain. This is true here also. So we still need to derive the contrasts repeated below.

(10) Jo called two people. Therefore, she called ✓less than three / # at most two.

(11) Jo called ✓less than three / # at most two people, but not one.

From *or*/*some* NP_{SG} we learned that contrasts like this can be derived by considering exhaustification relative to just a natural subclass of the DA, and its outcome. Below we explore this for CMNs/SMNs also. As for *or*/*some* NP_{SG}, this can only be meaningfully done for a domain with a minimum of 3 elements, that is, for a CMN/SMN pair such as *less than three* / *at most two*, with alternatives as in (58).

First, consider exhaustification relative to pre-exhaustified *non-singleton* DA and σA, $\mathrm{O}_{\mathrm{ExhSgDA}+\sigma\mathrm{A}}$. Assume that pre-exhaustification of each SgDA happens relative to all the other SgDA.

(71) $\mathrm{O}_{\mathrm{ExhSgDA}+\sigma\mathrm{A}}\Box_S(0\vee 1\vee 2)$

a. $\Box_S(0\vee 1\vee 2)\wedge$

b. $\neg \underbrace{\underbrace{\mathrm{O}\Box_S 0}_{\Box_S 0\wedge\neg\Box_S 1\wedge\neg\Box_S 2}}_{\Box_S 0\rightarrow\Box_S 1\vee\Box_S 2} \wedge\neg \underbrace{\underbrace{\mathrm{O}\Box_S 1}_{\Box_S 1\wedge\neg\Box_S 0\wedge\neg\Box_S 1}}_{\Box_S 1\rightarrow\Box_S 0\vee\Box_S 2} \wedge\neg \underbrace{\underbrace{\mathrm{O}\Box_S 2}_{\Box_S 2\wedge\neg\Box_S 0\wedge\neg\Box_S 1}}_{\Box_S 2\rightarrow\Box_S 0\vee\Box_S 1} \wedge$

c. $\neg\Box_S 0\wedge\neg\Box_S(0\vee 1)$

(M1) no ignorance / 'all winners':
$\Box_S 0\wedge\Box_S 1\wedge\Box_S 2$ ✗
(Impossible because of the nature of the domain.)

(M2) partial ignorance with positive certainty / 'one winner':
$\Box_S 0\wedge\cancel{\neg\Box_S}\Box_S\neg 1\wedge\cancel{\neg\Box_S}\Box_S\neg 2$ ✗
(The first implication would end up false.)

(M3) partial ignorance with negative certainty / 'one loser':
$\Box_S\neg 0\wedge\neg\Box_S 1\wedge\neg\Box_S 2$ ✗/✓
(For $\Box_S\neg 2$, in conjunction with the prejacent, clash with the σA-implicature $\neg\Box_S(0\vee 1)$. Can be fixed via σA-pruning, as discussed for (68).)

(M4) total ignorance / 'no winner':
$\neg\Box_S 0\wedge\neg\Box_S 1\wedge\neg\Box_S 2$ ✓

To sum up, the ExhSgDA-implicatures are compatible with partial ignorance of the negative certainty 'one loser' type and with total ignorance.

Second, consider exhaustification relative to pre-exhaustified *non-singleton* DA and σA, $\mathrm{O}_{\mathrm{ExhNonSgDA}+\sigma\mathrm{A}}$. Assume that pre-exhaustification of each NonSgDA happens relative to all the other NonSgDA.

(72)
$\mathrm{O}_{\mathrm{ExhNonSgDA}+\sigma\mathrm{A}}\Box_S(0\vee 1\vee 2)$

a. $\Box_S(0\vee 1\vee 2)\wedge$

b. $\neg \underbrace{\underbrace{\mathrm{O}\Box_S(0\vee 1)}_{\Box_S(0\vee1)\wedge\neg\Box_S(0\vee2)\wedge\neg\Box_S(1\vee2)}}_{\Box_S(0\vee1)\rightarrow\Box_S(0\vee2)\vee\Box_S(1\vee2)} \wedge\neg \underbrace{\underbrace{\mathrm{O}\Box_S(0\vee 2)}_{\Box_S(0\vee2)\wedge\neg\Box_S(0\vee1)\wedge\neg\Box_S(1\vee2)}}_{\Box_S(0\vee2)\rightarrow\Box_S(0\vee1)\vee\Box_S(1\vee2)} \wedge\neg \underbrace{\underbrace{\mathrm{O}\Box_S(1\vee 2)}_{\Box_S(1\vee2)\wedge\neg\Box_S(0\vee1)\wedge\neg\Box_S(0\vee2)}}_{\Box_S(1\vee2)\rightarrow\Box_S(0\vee1)\vee\Box_S(0\vee2)} \wedge$

c. $\neg\Box_S 0\wedge\neg\Box_S(0\vee 1)$

(M1) no ignorance / 'all winners':
$\Box_S 0 \wedge \Box_S 1 \wedge \Box_S 2$ ✗
(Impossible because of the nature of the domain.)

(M2) partial ignorance with positive certainty / 'one winner':
$\Box_S 0 \wedge \cancel{\neg\Box_S}/\Box_S\neg 1 \wedge \cancel{\neg\Box_S}/\Box_S\neg 2$ ✗/✓
(Clash with the σA-implicature $\neg\Box_S 0$. Can be fixed via σA-pruning, as discussed for (68).)

(M3) partial ignorance with negative certainty / 'one loser':
$\Box_S\neg 0 \wedge \neg\Box_S 1 \wedge \neg\Box_S 2$ ✗
(Consider the third implication. Suppose $\Box_S\neg 0$ is true. If $\neg\Box_S 1 \wedge \neg\Box_S 2$ is true also, then the whole consequent is false, so for the implication to be true, the antecedent $\Box_S(1 \vee 2)$ must be false. But this would contradict the conjunction of the prejacent $\Box_S(0 \vee 1 \vee 2)$ with our assumption $\Box_S\neg 0$, which entails $\Box_S(1 \vee 2)$.)

(M4) total ignorance / 'no winner':
$\neg\Box_S 0 \wedge \neg\Box_S 1 \wedge \neg\Box_S 2$ ✓

To sum up, the ExhNonSgDA-implicatures are compatible with partial ignorance of the positive certainty 'one winner' type and with total ignorance.

Third, and last, consider exhaustification relative to all the pre-exhaustified DA (that is, the full set of DA, both singleton and non-singleton) and σA, $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}$. If we continue to assume that pre-exhaustification of the SgDA and NonSgDA happens as before, that is, relative to all the other SgDA and NonSgDA, respectively, the result in this case is simply the intersection of the results we obtained in the $\mathrm{O}_{\mathrm{ExhSgDA}+\sigma\mathrm{A}}$ and $\mathrm{O}_{\mathrm{ExhNonSgDA}+\sigma\mathrm{A}}$ case, that is, total ignorance.

The solution to the difference with respect to the strength of the ignorance effect in CMNs/SMNs that we will propose is then as follows: By default both CMNs and SMNs have to be exhaustified relative to the full set of ExhDA and σA, and the result is as in the no pruning case — total ignorance. In the presence of a context of partial ignorance of the 'one winner' or 'one loser' type, CMNs but not SMNs are able to also prune their DA set to either just the NonSgDA or just the SgDA to accommodate those meanings. We have now captured (10)-(11).

But this proposal seems to suggest that CMNs can accommodate certainty only when prompted by explicit context. Yet many of the common examples used in the literature to argue for a contrast between them are seemingly out-of-the-blue, like the ones below. How do we capture these examples?

(73) [Parent:] I have ✓more than two / ?? at least three children.

(74) [Flight attendant:] This plane has more than five / ?? at least six emergency exits.

Note that in both these examples the reason why the SMN is found to be odd is only because it comes from the parent / flight attendant, thus, a person assumed to be in the know, and it wouldn't necessarily be found odd if uttered by someone else. I will argue that oddness here comes from a clash of SMNs with an implicit assumption of knowledge, thus, with an implicit 'winner' context. Then the fact that CMNs can accommodate it but SMN can't is precisely what we would expect.

The general account then works as described. Moreover, just as for *or*/*some* NP_{SG}, due to the similar logical shape, this account predicts parallel total and partial variation effects under an overt necessity modal or in the scope of a universal quantifier also. This prediction seems to be borne out. (One difference is that variation, whether total or partial, can be of two types, coming from $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}} > \Box/\forall > \vee$ or $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}} > \Box_{\mathrm{S}} > \vee > \Box/\forall$; the former/latter for $\Box$ for SMNs has already studied under the label the 'authoritative'/'speaker insecurity' reading of SMNs under necessity modals, cf. Büring 2008 and literature since then; the latter for $\forall$ for CMNs/SMNs is aka as the quantificational variability effect of CMNs/SMNs under a universal quantifier, corroborated experimentally by Alexandropoulou et al. 2015.)

(75) a. Jo must call less than three / at most two people. ($\rightsquigarrow$ variability)
b. Jo must call two people. Therefore, she must call ✓less than three / # at most two.
c. Jo must call ✓less than three / # at most two people, but not one.

(76) a. Everyone called less than three / at most two people. ($\rightsquigarrow$ variability)
b. Everyone called two people. Therefore, everyone called ✓less than three / # at most two.
c. Everyone called ✓less than three / # at most three people, but not one.

Before we wrap up, one final issue. This is an issue that we left open at the end of Section 3.3, and it had to do with the fact that in a positive context the strong σA-implicatures of CMNs/SMNs yielded an undesirable 'exactly' implicature, (61). How does this result interact with the ExhDA-implicatures that we have been studying in this section? Do we now have a way to rule it out?

While strong σA-implicatures are σA-implicatures computed below $\Box_{\mathrm{S}}$, we know that $\mathrm{O}_{\mathrm{ExhDA}}$ is successful only above $\Box_{\mathrm{S}}$, so the exhaustification configuration we must study is a split configuration, as below. In this exhaustification $\mathrm{O}_{\mathrm{ExhDA}}$ asserts its prejacent, which contains $\mathrm{O}_{\sigma\mathrm{A}}$, (77-a), and negates the ExhDA, which should technically also contain $\mathrm{O}_{\sigma\mathrm{A}}$, but since it is vacuous (an ExhDA, just like a DA, doesn't have an σA of its own) we left it out, (77-b).

(77) John called less than three people / at most two people.
$\mathrm{O}_{\mathrm{ExhDA}}(\Box_S \mathrm{O}_{\sigma\mathrm{A}}(0 \vee 1 \vee 2))$
a. $\Box_S \mathrm{O}_{\sigma\mathrm{A}}(0 \vee 1 \vee 2) \wedge$
b. $\neg\mathrm{O}\Box_S 0 \wedge \neg\mathrm{O}\Box_S 1 \wedge \neg\mathrm{O}\Box_S 2 \wedge \neg\mathrm{O}\Box_S(0 \vee 1) \wedge \neg\mathrm{O}\Box_S(1 \vee 2) \wedge \neg\mathrm{O}\Box_S(0 \vee 2)$

$$= \underbrace{\underbrace{\overbrace{\Box_S((0\vee 1\vee 2)\wedge\neg(0\vee 1))}^{(a)}}_{=\Box_S 2} \wedge \overbrace{\neg\Box_S 0\wedge\neg\Box_S 1\wedge\neg\Box_S 2}^{(b)}}_{\perp}$$

As we can see, the assertion yields certainty about the domain, $\Box_S(0 \vee 1 \vee 2)$. Then $\mathrm{O}_{\sigma\mathrm{A}}$ strengthens it to certainty about, essentially, a singleton DA, $\Box_S 2$. But $\mathrm{O}_{\mathrm{ExhDA}}$ yields total ignorance (the default result in default contexts), including ignorance about this singleton DA, $\neg\Box_S 2$. The result is a contradiction. This means that the illicit 'exactly' σA-implicature is in fact not generated, solving (61).

At the same time, as also mentioned in Section 3.3, in (65), this doesn't mean that CMNs/SMNs in a positive context can never get a strong σA-implicature parse — if we prune the σA set and negate the next strongest σA, the resulting σA-implicature is perfectly plausible. This is also as we would expect: As already mentioned, there are reasons to believe that, if an σA-implicature of a CMN/SMN clashes with the ExhDA-implicatures, the clash is always fixed by pruning the offending σA.

Now, we don't want to say that σA-pruning happens for CMNs/SMNs only as a way to accommodate an explicitly contradictory context or to fix a clash with the ExhDA-implicatures. Otherwise, we would expect an utterance of *less than 20* to always by default generate an implicature that *not less than 18*. We will assume, as is often done in the literature, that the σA of CMNs/SMNs are in general quite sensitive to issues related to context, including granularity (cf. also shown by Cummins et al. 2012). For example, *Jo called more than two people* might generate ignorance about 3, 4, or 5, but not necessarily about 10, and quite surely not about 25 or a hundred. Our discussion of σA-implicatures alongside with ignorance has thus proven fruitful in shedding new light not only on σA-implicatures but also on ignorance.

Last but not least, note that, because our solution for ignorance depends on DA and BNs only have σA, we predict them not to be able to give rise to ignorance, (78), and in fact to be incompatible with it, (79) (unless we use special adjustment mechanisms, e.g., emphasize: *but she does have three*). This seems to be borne out.

(78) (How many people did Jo call?) Three. ($\not\rightsquigarrow$ ignorance)

(79) I don't know how many people Jo called, but she called # three / ✓more/less than three / ✓at least/most three.

3.5 Polarity sensitivity

The recipe we used to capture the facts for *or/some* NP_{SG} has already helped us capture ignorance in CMNs/SMNs. Will it help us with polarity sensitivity as well?

(12) Jo didn't call ✓less than two people / # at most one person.

(13) If Jo called ✓less than two people / ✓at most one person, she won.

(14) Everyone who called ✓less than two people / ✓at most one person won.

Let's first observe exhaustification across negation.

(80) $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}(\neg(0\vee 1))$

a. $\neg(0\vee 1)$

b. $\neg \underbrace{\mathrm{O}(\neg 0)}_{\substack{\neg 0\wedge\neg\neg 1,\ =\neg 0\wedge 1 \\ \text{already excluded by the prejacent}}} \wedge\neg \underbrace{\mathrm{O}(\neg 1)}_{\substack{\neg 1\wedge\neg\neg 0,\ =\neg 1\wedge 0 \\ \text{already excluded by the prejacent}}}$

c. $\underbrace{\neg \underbrace{(\neg(0\vee 1\vee 2))}_{\text{not entailed by the prejacent}}}_{0\vee 1\vee 2}$

As for *or/some* NP_{SG}, each of the ExhDA is incompatible with the prejacent, therefore already excluded by it, so negating them does not lead to any strengthening. The σA however is different — while for *or/some* NP_{SG} it was also entailed by the prejacent, here it is not, and together with the prejacent gives rise to an illicit 'exactly two' meaning. But the solution for *or/some* NP_{SG} didn't depend on the outcome from σA-implicatures, so for now we will put them aside and retain only that $\mathrm{O}_{\mathrm{ExhDA}}$ of a CMN/SMN across negation does not lead to proper strengthening here either.

Now let's observe what happens when we exhaustify CMNs/SMNs across *if/every*. As for *or/some* NP_{SG}, let's assume again that exhaustification across *if/every* takes into account the presupposition-enriched content of the prejacent and of the alternatives. Thus, it has the shape below (where again v = the world/individual variable *w/x* from the conditional / universal, 0 = 'Jo called 0 people / 'called 0 people', and b = 'Jo called 1 person / 'called 1 person').

(81) $\mathrm{O}^{\mathrm{S}}_{\mathrm{ExhDA}+\sigma\mathrm{A}}\forall v[(0\vee 1)_v \rightarrow W_v]$

a. $\forall v[(0\vee 1)_v \rightarrow W_v]\wedge\exists v[(0\vee 1)_v]\wedge$

b. $\underbrace{\neg \underbrace{\mathrm{O}(\forall v[0_v \rightarrow W_v]\wedge\exists v[0_v])}_{(\forall v[0_v\rightarrow W_v]\wedge\exists v[0_v])\wedge\neg(\forall v[1_v\rightarrow W_v]\wedge\exists v[1_v])}}_{(\forall v[0_v\rightarrow W_v]\wedge\exists v[0_v])\rightarrow(\forall v[1_v\rightarrow W_v]\wedge\exists v[1_v])} \wedge\neg \underbrace{\underbrace{\mathrm{O}(\forall v[1_v \rightarrow W_v]\wedge\exists v[1_v])}_{(\forall v[1_v\rightarrow W_v]\wedge\exists v[1_v])\wedge\neg(\forall v[0_v\rightarrow W_v]\wedge\exists v[0_v])}}_{(\forall v[1_v\rightarrow W_v]\wedge\exists v[1_v])\rightarrow(\forall v[0_v\rightarrow W_v]\wedge\exists v[0_v])}$

c. $\neg(\forall v[(0 \vee 1 \vee 2)_v \rightarrow W_v] \wedge \exists v[(0 \vee 1 \vee 2)_v \rightarrow W_v]$

(M1) (a) $\wedge \exists v[0_v] \wedge \exists v[1_v]$ (cf. $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}(\Diamond(0 \vee 1)))$
'…and there is a world where Jo called 0 / individual who called 0 and there is a world where Jo called 1 / individual who called 1.'

(M2) (a) $\wedge \neg\Box\exists v[0_v] \wedge \neg\Box\exists v[1_v]$ (cf. $\mathrm{O}_{\mathrm{ExhDA}+\sigma\mathrm{A}}(\Box_{\mathrm{S}}(0 \vee 1)))$
'…and the speaker is ignorant/indifferent whether there is a world where Jo called 0 / individual who called 0 and ignorant/indifferent whether there is a world where Jo called 1 / individual who called 1.'

Again, aside from the σA-implicature — which this time is a plausible indirect scalar implicature — the result is exactly as in the *or/some NP$_{SG}$* case: $\mathrm{O}_{\mathrm{ExhDA}}$ of a CMN/SMN across *if/every* does lead to proper strengthening.

As for *or/some NP$_{SG}$*, we will propose that both CMNs and SMNs undergo exhaustification in the strong sense, but SMNs additionally carry a Proper Strengthening requirement. Given that this requirement cannot be satisfied under negation but it can be under *if/every*, this captures their distribution in (12)-(14).

Finally, as for *or/some NP$_{SG}$*, we may wonder if $\mathrm{O}^{\mathrm{S}}_{\mathrm{ExhDA}}$can take into account non-truth-conditional content other than presuppositions, for example, implicatures. Again we can check by trying to embed CMNs/SMNs under a downward-entailing operator with a positive implicature such as *few*. If $\mathrm{O}^{\mathrm{S}}_{\mathrm{ExhDA}}$took into account this implicature, this should lead to proper strengthening, and SMNs should be felicitous. However, they are not, suggesting that, just as for *some NP$_{SG}$*, $\mathrm{O}^{\mathrm{S}}_{\mathrm{ExhDA}}$ in this case also takes into account only presuppositions.

(82) Few people believe that Mary is ✓more than 19 / # at least 20 years old.

Before we wrap up, one final issue. This is the second issue that we left open at the end of Section 3.3, and which also came up above; it has to do with the fact that strong $\mathrm{O}_{\sigma\mathrm{A}}$ across negation yielded an undesirable 'exactly' implicature for all of BNs, CMNs, and SMNs, (64). Can we shed any new light on this?

The solution I propose is as follows: I propose that $\mathrm{O}_{\sigma\mathrm{A}}$ of a BN/CMN/SMN across negation in fact proceeds relative to two types of σA. First, the traditional σA, of the form *not m*. Second, a new set of σA obtained by deleting the negation, thus, of the form *m*. Since these two types of alternatives contradict each other, the result is a clash, and so no illicit 'exactly' implicature is in fact generated. I illustrate below for *three* / *more than three* / *at least three*.

(83) Jo didn't call three / more than two / # at least three people.
$\mathrm{O}_{\sigma\mathrm{A}}\neg(3 \vee 4 \vee \ldots)$
a. $\neg(3 \vee 4 \vee \ldots) \wedge$

b. $\neg\neg(2\vee\ldots)\wedge\neg\neg(1\vee\ldots)\wedge\ldots$ (traditional σA)
c. $\neg(2\vee\ldots)\wedge\neg(1\vee\ldots)\wedge\ldots$ (new σA, obtained by deletion of $\neg$)
$=\bot$

However, if we exhaustify across $\Box_S$ — our usual exhaustification rescuing operator — the result is consistent and yields ignorance.

(84) Jo didn't call three / more than two / # at least three people
$O_{\sigma A}\Box_S\neg(3\vee 4\vee\ldots)$
a. $\Box_S\neg(3\vee 4\vee\ldots)\wedge$
b. $\neg\Box_S\neg(2\vee\ldots)\wedge\neg\Box_S\neg(1\vee\ldots)\wedge\ldots$ (traditional σA)
c. $\neg\Box_S(2\vee\ldots)\wedge\neg\Box_S(1\vee\ldots)\wedge\ldots$ (new σA, obtained by deleting $\neg$)
'In all the worlds compatible with what the speaker believes the relevant number is not three or more but the speaker is not sure which one of the remaining numbers it is.'

This new proposal for $O_{\sigma A}$ takes care of our second illicit 'exactly' σA-implicatures puzzle and moreover captures an intuitively correct ignorance effect of scalar items under negation. At the same time, note that it doesn't affect our solution for polarity sensitivity at all. While, due to the nature of the σA, the insertion of $\Box_S$ makes a difference to the result of $O_{\sigma A}$ across negation, it does not make any difference to the result of O_{ExhDA} across negation — the ExhDA $O\Box_S\neg 0$ and $O\Box_S\neg 1$ continue to be each already excluded by the prejacent, so their negation cannot lead to proper strengthening.

Last but not least, note that, because our solution for anti-negativity depends on DA, and because our BNs only have σA, we predict them not to be able to give rise to anti-negativity. Insofar as I know, there is indeed no language where the BN exhibits anti-negativity, so this too seems to be borne out.

3.6 Comparison to previous literature

To my knowledge, none of the existing literature covers all of our starting patterns for CMNs/SMNs, so in that sense the proposal above already represents empirical progress.

At the same time, there have been numerous other solutions to ignorance and polarity sensitivity in CMNs/SMNs in the literature, including approaches based on alternatives and exhaustification. So again we will ask: Is our choice of a solution merely a matter of theoretical preference, or are there empirical advantages as well?

First, we proposed new truth conditions for CMNs/SMNs that marry existing insights with a new decomposition of the modifiers. The result is that we don't just capture the same bounding entailments as the existing literature, but also make sense of how they arise from their morphological pieces. Though likely unfaithful in the details, I believe this decomposition is faithful in the essentials — for example, an adaptation of Chen 2018's recent careful decomposition of *at*-superlatives to SMNs reveals the same, namely, that they make reference in their truth conditions to a domain parasitic on the scalar element. And, crucially, it gives us a way to straightforwardly capture the intuitive similarity between not just SMNs but also CMNs to not just disjunction but also indefinites — formally, they all make reference in their truth conditions to a domain.

Second, we share with most of the existing alternative-based proposals for ignorance the use of alternatives that are symmetric, that is, alternatives that in the plain case (no intervening operators) can't be excluded together without contradiction. However, the similarity stops here. As the existing literature shows, there are very many different ways to generate such alternatives, and each has non-trivial consequences, leading to sometimes quite different theoretical commitments and empirical consequences (Büring 2008, Mayr 2013, Kennedy 2015, Spector 2015, Schwarz 2016). On our approach the alternatives are derived in a fully general way (and in contrast to some of the literature, for all of *more/less than*, *at least/most*), as for disjunction/indefinites, by replacement of the domain/scalar item with its subsets/scalemates. The fact that we derive them not just for SMNs but also CMNs helps us capture a fact neglected by most of the literature so far but which is beginning to be recognized, namely, the fact that CMNs can give rise to ignorance / other variation effects also (Westera & Brasoveanu 2014, Alexandropoulou et al. 2015, Cremers et al. 2017, Nouwen et al. 2018); at the same time, our insights from *or*/*some* NP_{SG} give us a way to explain how this effect may be weaker. And our more general conception of $\Box$ helps us capture the fact that the modal effect in seemingly episodic contexts can be not just ignorance, but also indifference. Finally, our use of ExhDA can also help shed new light on longstanding puzzles regarding embedding under possibility modals, as discussed in Fn. 8.

Third, there isn't currently any solution for all of our starting patterns for CMNs and SMNs in downward-entailing environments. The most comprehensive proposal, due to Cohen & Krifka (2011, 2014), and still limited to SMNs, is as follows: SMNs are ambiguous between a meaning where they acquire their truth conditions via implicature, and which is therefore unavailable in downward-entailing environments, and an evaluative meaning, which is fine in downward-entailing environments and in particular thrives in environments such as the antecedent of a conditional / restriction of a universal. However, as the authors acknowledge, an *at*-superlative is bad under negation even when used evaluatively, and for this they have no answer. Our solution not only captures the contrast between acceptability in the scope of negation vs. presuppositional downward-entailing environments but does so in a way that fully reflects the similarity of this contrast to contrasts reported for disjunction/indefinites, and also makes sense of the patterns for CMNs. (Our solution for anti-negativity is similar to Spector 2015's $O_{\text{IE-DA}}$+ban on vacuous exhaustification solution for *at least*, but uses more principled DA, includes *at most* and CMNs, and has the advantages of not using $O_{\text{IE-DA}}$+ban on vacuous exhaustification, with better empirical predictions, as discussed in Section 2.5, including for CMNs.)

Last but not least, in contrast to all the most recent approaches to BNs or CMNs/SMNs, we restore their classical σA-implicatures, show how the original issues related to them can be addressed, and also how their reintroduction makes welcome predictions for a general theory of numerals as scalar items, in a variety of ways.

3.7 Summary

In this section we used our recipe for capturing similarity and variation with respect to ignorance and polarity sensitivity in *or*/*some* NP_{SG} to capture the same effects in CMNs and SMNs also. We noticed that, once we decomposed the modifiers to uncover not just the scalar element but also the domain in their truth conditions, not only the ignorance and polarity sensitivity patterns of CMNs/SMNs, but also their scalar implicature patterns (including new solutions to old puzzles) naturally follow. The result is a theory of bare and modified numerals that improves on existing results in multiple ways, not only through greater theoretical generality but also through better and broader empirical coverage.

4 Conclusion and outlook

We started from patterns showing that the pairs *or*/*some* NP_{SG} and CMNs/SMNs are, given the same domain of people/degrees, truth conditionally equivalent, but then contrast with respect to ignorance and polarity sensitivity. This variation is intriguing for each individual pair, but it is particularly interesting when we bring the pairs together, as it points to a remarkable similarity of these effects across disjunction/indefinites and modified numerals. Building on Chierchia (2013)'s approach to epistemic indefinites and polarity sensitivity, we proposed a fully general and unified account of ignorance and polarity sensitivity in these items as follows: All these items make reference in their truth conditions to both a domain and a scalar item, and these naturally generate subdomain and scalar alternatives. Alternatives are factored into meaning via silent exhaustivity operators — we have assumed exhaustification with O. Items may vary with respect to whether their subdomain alternatives must be used as derived or rather pre-exhaustified, or with respect to which alternatives they activate by default — we have assumed that all our items are by default exhaustified relative to pre-exhaustified subdomain alternatives and scalar alternatives. Without any intervening operator, this exhaustification fails; however, with an intervening possibility or necessity modal, it leads to a Free Choice effect. An occurrence of these items in a seemingly episodic context is actually an occurrence under a null speaker-oriented epistemic (or agent-oriented bouletic) necessity modal, so exhaustification yields an epistemic (bouletic) Free Choice effect = ignorance (indifference). In the default case this effect is total; compatibility with certainty comes from an item's lexically encoded, parametric ability to prune a natural subclass of its subdomain alternatives. An occurrence of these items in a downward-entailing environment such as the scope of negation (but not the antecedent of a conditional or the restriction of a universal) is an occurrence where exhaustification cannot lead to a properly stronger meaning; anti-negativity comes from an item's lexically encoded, parametric inability to tolerate such a result. This proposal not only offers a fully unified solution for similarity and variation with respect to ignorance and polarity sensitivity in disjunction/indefinites and modified numerals, but also leads to a better understanding of disjunction/indefinites and especially numerals more generally.

There are of course many open issues.

First, this discussion raises questions about further empirical patterns. As we already cursorily did, one may wonder about variation in the Free Choice effect with possibility modals, or variations in polarity sensitivity effects with other downardentailing operators. Also, other patterns of interest concern embedding under a downward-entailing operator at a distance, embedding under a downward-entailing operator with an intervening modal or factive, embedding under multiple downwardentailing operators (cf. in the literature on PPIs: Szabolcsi 2004, Nicolae 2012, Spector 2014, Nicolae 2017; in the literature on CMNs/SMNs: Mihoc & Davidson 2017 for experimental testing of some of these patterns), or embedding in the antecedent of a conditional / restriction of a universal contingent on the polarity of the modified numeral, of the predicate it combines with, of the antecedent/restriction, or of the continuation (cf. in the literature on SMNs: Nilsen 2007, Cohen & Krifka 2011, 2014 and Mihoc & Davidson 2017 for experimental testing of some of these patterns; in the literature on NPIs: Crnič 2011, who also provides suggestions for how to handle such effects on an alternatives-and-exhaustification approach).

Second, this proposal makes predictions but also raises questions for the range of empirical variation. Our (following Chierchia 2013) parametric approach to variation in ignorance and polarity sensitivity in *or*/*some* NP_{SG} predicts that there could be a variant of *or* with anti-negativity, or a variant of *some* NP_{SG} incompatible with positive certainty and without anti-negativity. Both these predictions are borne out, as we know from the existing discussions of the French disjunction *soit ... soit* (Spector 2014) or the French disjunction *ou* (Nicolae 2017), or the German indefinite *irgendein* (Chierchia 2013 and references therein). But it also predicts that there could be a variant of *or* without total ignorance, or a variant of CMNs/SMNs where they are like SMNs/CMNs. Given the literature so far, I am not sure if such variants exist; if it turns out that they don't, then we will have to find a way to derive rather than leave free their parametric setting.

Last but not least, this proposal raises new questions regarding the nature of ungrammaticality. How do violations of no DA-pruning and proper strengthening compare to logical contradiction, cancelation of scalar implicatures, or logical redundancy? This question has already been asked for ignorance in CMNs/SMNs (Geurts et al. 2010), but in light of the discussions here (which showed the same effects in disjunction/indefinites, but also that the penalty can vary by item) it can be asked again, and one may now wonder about polarity sensitivity also.

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